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**Abstract**

This paper reviews the use of Fourier transform methods in the pricing of contingent claims. This is a very promosing topic in finance, given the scarcity of closed-form solutions for derivative prices. It is shown that solving for the Fourier Transform is much easier than solving for the price, especially so under complex probability models, such as affine jump diffusions. In fact, explicit solutions for the Fourier transform are guaranteed for many types of contingent claims. As a consequence, the only remaining numerical issue in that case is the inversion of the Fourier transform.

**Résumé**

Dans ce papier nous passons en revue la méthode de transformée de Fourier tels qu'elle est utilisée pour l'évaluation des biens contingents. Cette voie de recherche est encore prometteuse étant donnée la rareté des formes explicites pour les prix de certains produits dérivés. En fait, pour les modèles probabilistes riches comme les processus affines avec sauts, il est plus facile de résoudre pour la transformée de Fourier du prix que le prix lui même. On constate a cet effet que plusieurs biens contingents permettent d'avoir la transformée de Fourier sous une forme explicite et pour trouver le prix, il suffit de l'inverser, très souvent numériquement.

# 1 Introduction

Fourier transform methods have recently become increasingly popular among both academics and practitioners. One of the main reasons for their success is the speed of implementation, as compared to other numerical methods such as Monte Carlo simulation or finite differences. The popularity of the Fourier transform approaches is not limited to finance. In fact, it is deeply rooted in mathematical practice, where mathematicians used to apply Fourier techniques whenever a direct solution to their problem was not feasible. The principle behind the method is simple: take the problem to the "Fourier world" to seek a solution, then bring the solution back to the "real world". Going back and forth between the real and Fourier world is carried-out through Fourier transforms, which reduces to the evaluation of two integrals. The method is computationally efficient when the first integral (that corresponds to the Fourier world) is available in closed-form.

Option pricing was for a long time considered to be a difficult problem. With the introduction of more sophisticated probability models, such as Lévy processeso or stochastic volatility models, option pricing has become even more challenging. The first paper introducing Fourier technique to finance was Heston [6]. This work was followed by a series of papers from Scott [12], Carr and Madan [14], Bakshi et al. [7], Dempster and Hong [4], and Liu et al. [13], to name a few. Cerny [15] provided a clean introduction to the method, with a focus on the numerical Fast Fourier Transform (FFT) algorithm. Although the purpose of each of these papers was to come up with a closed-form solution for the price, the applicability of Fourier transform was limited to the scope of the specific application considered. The versatility of the technique was fully exposed in the pioneering works of Bakshi and Madan [8] and Duffie et al. [2].

In Duffie et al. [2], the authors propose a two-step process: first, the Fourier transform of the price is computed, then an inversion technique is applied to recover the original price. Because many options have similar payoffs, up to adjusting some parameters, this allows the nesting of many results into a single valuation equation. The most important result though is due to Bakshi and Madan [8], which is to some extent a generalization of all previous papers. The authors show that analyticity of the joint characteristic function of all uncertainties in the economy is sufficient to handle many pricing problems. It that case, differentiating or translating (or both) that function allows to switch between many option prices. The core advantage of Fourier transform methods, as depicted in that paper, is that the solution to the joint characteristic function is easy to obtain in many instances, owing to its smoothness. More precisely, to obtain a closed-form solution for the price of a contingent claim, one has to solve partial (integro) differential equations, which are notoriously difficult to handle because of the presence of indicator functions in the boundary conditions. On the other hand, since the characteristic function is infinitely differentiable (so long as the moments are defined), its partial differential equation is relatively easy to solve.

This paper's intention is to overview what has been achieved so far in option pricing using Fourier transform, focusing mainly on the papers by Bakshi and Madan [8] and Duffie et al. [2]. It is organized as follows. In the next section, general results are stated and proved. The most difficult side in this method is to come up with an explicit expression for the Fourier transform. Under some assumptions and technical conditions, this transform is linked to the characteristic function of the uncertainty. Section 2 shows how to apply the methodology to some specific contingent claims, most of them with nonlinear (unidimensional) payoffs. Section 3 provides an example of how the methodology can be extended to multidimensional payoffs. Section 4 shows how to solve explicitly for the analytic Fourier transform in a number of interesting models, some of them already proposed in the literature. Section 5 provides closed-form solutions for some specific stochastic processes and reports on numerical results. Section 6 concludes.

# 2 General Set-up

## 2.1 The Duffie, Pan and Singletton approach

We consider an economy where uncertainty is driven by $k$ state variables stacked in a vector $X$ living in a complete filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, P)$, where $\boldsymbol{F}$ is the information filtration satisfying the usual assumptions as stated in Protter [1], (Definition 1, page3). We suppose a terminal date $T$ where all activities in the economy cease, so that $\mathcal{F} \overset{def}{=} \mathcal{F}_T$. We assume as in Duffie et al. [2] that the state vector is Markov in some state space $D \subset \mathbb{R}^k$ and satisfies the following stochastic differential equation (SDE henceforth),[1]

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dZ_t \tag{1}$$

where $W$ is a standard $\boldsymbol{F}$ Brownian motion in $\mathbb{R}^k$, $\mu : D \to \mathbb{R}^k$, $\sigma : D \to \mathbb{R}^{k\times k}$ and $Z$ is a pure jump process characterized by a jump size distribution $\nu$ in $\mathbb{R}^k$ and jump arrival intensity $\lambda(X_t)$ for some given function $\lambda : D \to [0,\infty)$. It is assumed throughout that $\mu$, $\sigma$, $\nu$, and $\lambda$ are well-behaved so that a solution to (1) exists. The process (1) can be seen as the time nonhomogeneous version of the Compound Poisson Process, where conditional on the path of $X$, $Z$ is a Poisson process with intensity $\lambda(X_t)$.

For the purpose of pricing contingent claims, we suppose that the economy is exempt from arbitrage opportunities, so that a pricing functional is defined as follows:

$$p_t = E^Q\left[\exp\left(-\int_t^T r(X_s)\,ds\right)\chi(X_T)\right] \tag{2}$$

where $p_t$ is the no-arbitrage price, $r : D \to [0,\infty)$ is the short rate process in the economy and $\chi$ is the payoff function. The expectation is computed with respect to an equivalent probability measure $Q$. This measure has the property that discounted prices by the numéraire process

$$\delta_t \equiv \exp\left(-\int_0^t r(X_s)\,ds\right)$$

are martingales. Since markets might be incomplete and hence multiple martingale measures could exist, it is not necessary for our purpose to specify which measure we are referring to. We suppose that the choice has already been made.

For many interesting applications, the payoff function has the following form,

$$\chi(X) = (v_0 + v_1^\intercal X)\left(\exp(u^\intercal X) - K\right)\mathbf{1}_{\{w^\intercal X \in \Lambda\}} \tag{3}$$

where $v_0$ is a scalar, $v_1$ , $u$, and $w$ are $k\times 1$ vectors of constants (or complex vectors in some cases), $K$ is a constant, and $\Lambda$ is a Borel set. For instance, the price of a zero-coupon bond or the price of a corporate bond (See Duffie et al. [2] for an example) can be recovered using (2)-(3) by setting $v_0 = w = 1$, $v_1 = u = 0$, and $\Lambda = \mathbb{R}$. The payoff of a plain vanilla European call (*resp. put*) option is obtained by setting $v_0 = 1$ (resp. $-1$), $v_1 = 0$, $u = w$, and $\Lambda = \{w^\intercal X \geq K\}$ (*resp.* $\Lambda = \{w^\intercal X \leq K\}$) where $K$ is the strike price. To further stress the generality of equation (3), the payoff of a call option on a zero-coupon yield whose payoff is of the form $\max(b^\intercal X - K, 0)$ is obtained by setting $v_0 = -K$, $v_1 = w = b$, $u = 0$, and $\Lambda = \{b^\intercal X \geq K\}$. Further contingent claims will be explored in Section 5. Notice that Formulas (2)-(3) are extendable outside of pricing problems. For instance, the characteristic function, and hence the probability distribution of $X_t$, can be recovered by setting $r(\cdot) = 0$ in (2), $v_0 = v_1 = w = 1$, $u = i\theta$ (for $\theta \in \mathbb{R}^k$ and $i = \sqrt{-1}$) and $\Lambda = \mathbb{R}^k$ in (3); the conditional expectation will return the conditional characteristic function although the probability measure is changed.[2]

Our task now is to find an explicit expression for the Fourier transform of the price as given in (2). Before solving for the general case, we first focus on payoffs of the form $\exp(u^\intercal X)$. The contingent claim price is then given by

$$p_t = E^Q\left[\exp\left(-\int_t^T r(X_s)\,ds\right)e^{u^\intercal X}\middle|\,\mathcal{F}_t.\right] \tag{4}$$

Due to the difficulty of finding an explicit expression for the joint density of the state vector $X$, this expectation is not easy to compute. Proceeding differently, we can rely on the Feynmann-Kac result that establishes a link between conditional expectations and partial differential equations (PDE henceforth), to

[1] Note that from here onward, we will adopt many Duffie's et al. notation but in some cases we introduce our own.

[2] The expectation under the physical meausre can be computed by simply taking the parameters of that measure.

conjecture a solution for the PDE and reduce it to a set of ordinary differential equations (ODE). This approach was used by many authors to find explicit or quasi-explicit formulas for the prices of contingent claims (see for instance Heston [6], Bakshi et al. [8], Stein, and Stein [11] and Scott [12]). However, the type conjectured solutions is not arbitrary, but is tightly linked to the assumed functional form of the coefficients $\mu(X)$, $\sigma(X)$, $\lambda(X)$, and the jump size distribution $\nu(z)$. For this reason, we make the following simplifying assumption (Duffie et al. [2] page 1350)

- $\mu(X) = k_0 + k_1 \cdot X$, for $k_0 \in \mathbb{R}^k$, $k_1 \in \mathbb{R}^{k\times k}$,
- $\sigma(X)\sigma(X)^\intercal = H_0 + H_1 \cdot X$, for $H_0 \in \mathbb{R}^{k\times k}$, $H_1 \in \mathbb{R}^{k\times k\times k}$,
- $\lambda(X) = l_0 + l_1 \cdot X$, for $l_0 \in \mathbb{R}$, $l_1 \in \mathbb{R}^k$,
- $r(X) = \rho_0 + \rho_1 \cdot X$, for $\rho_0 \in \mathbb{R}$, and $\rho_1 \in \mathbb{R}^k$,
- $M_z(u) = \int_{\mathbb{R}^k} \exp(u^\intercal z)\, d\nu(z)$, the jump distribution transform, is well defined for $u \in \mathbb{C}^k$, where $\mathbb{C}^k$ is the set of $k$-tuples of complex numbers.

Under this assumption and some other regularity conditions, the solution to (4) has the following form:

$$\Psi(u, x, t, T) = \exp\left(\alpha(t) + \beta(t)^\intercal x\right) \tag{5}$$

subject to solving this set of ODE's:

$$\begin{aligned} \dot{\beta}(t) &= \rho_1 - k_1^\intercal \beta(t) - \frac{1}{2}\beta(t)^\intercal H_1 \beta(t) - l_1\left(M_z(\beta(t)) - 1\right) \\ \dot{\alpha}(t) &= \rho_0 - k_0^\intercal \beta(t) - \frac{1}{2}\beta(t)^\intercal H_0 \beta(t) - l_0\left(M_z(\beta(t)) - 1\right) \end{aligned} \tag{6}$$

with boundary conditions $\beta(T) = u$ and $\alpha(T) = 0$. Note that $u$ might be a $k$-tuple complex vector, so that the solution is complex valued. The following proposition justifies the conjecture in (5).

**Proposition 1** *(Duffie et al. page 1351) Suppose that:*

1. $E\left(\int_0^T |\gamma_t|\, dt\right) < \infty$ *where* $\gamma_t = \Psi_t\left(M_z(\beta(t)) - 1\right)\lambda(X_t)$*;*
2. $E\left(\left(\int_0^T \eta_t \cdot \eta_t dt\right)^{\frac{1}{2}}\right) < \infty$ *where* $\eta_t = \Psi_t \beta^\intercal(t)\sigma(X_t)$*; and*
3. $E\left(\left|\left(-\int_0^T r(X_s)\, ds\right)\Psi_T\right|\right) < \infty$*, then the solution to (4) is given by (5).*

**Proof. Proof.** The proof relies on Itô's lemma and absence of arbitrage. Treating the formula in (5) as a (complex-valued) price, we know by the very existence of a martingale measure in this economy that discounted prices have to be martingales. Let $\Psi_t^* = \exp\left(-\int_0^t r(X_s)\, ds\right)\Psi_t$ be the discounted price. From stochastic calculus, we have □

$$\Psi_t^* - \Psi_0^* = \int_0^t -r(X_s)\Psi_s^* ds + \int_0^t \exp\left(-\int_0^s r(X_u)\, du\right) d\Psi_s$$

As for $d\Psi_t$, we have from Protter [1] (Theorem 32, page 78) that

$$\begin{aligned} \Psi_t - \Psi_0 &= \int_0^t \left(\frac{\partial \Psi}{\partial s} + \frac{\partial \Psi}{\partial X}\mu(X_{s^-}) + \frac{1}{2}tr\left[\frac{\partial^2 \Psi}{\partial X \partial X^\intercal}\sigma(X_{s^-})\sigma(X_{s^-})^\intercal\right]\right) ds \\ &\quad + \int_0^t \frac{\partial \Psi}{\partial X}\sigma(X_{s^-})\, dW_s + \int_0^t \frac{\partial \Psi}{\partial X} dZ_s + \sum_{0<s\le t}\left(\Psi_s(X_s) - \Psi_s(X_{s^-}) - \frac{\partial \Psi}{\partial X}\Delta X_s\right) \\ &= \int_0^t \Psi_{s^-}\left(\begin{array}{c} \dot{\alpha}(s) + \dot{\beta}(s)^\intercal X_{s^-} + \beta(s)^\intercal \mu(X_{s^-}) + \frac{1}{2}tr\left[\beta(s)^\intercal \sigma(X_{s^-})\sigma(X_{s^-})^\intercal \beta(s)\right] \\ +\lambda(X_{s^-})\left(M_z(\beta(s)) - 1\right) \end{array}\right) ds \end{aligned}$$

$$+\int_0^t \Psi_{s^-}\beta(s)^\intercal \sigma(X_{s^-})\,dW_s + \int_0^t \Psi_{s^-}\left(e^{\beta^\intercal(s)z}-1\right)dN(s,z)$$
$$-\int_0^t \Psi_{s^-}\lambda(X_{s^-})\left(M_z(\beta(s))-1\right)ds$$

where these results have been used; $\Delta X_t = dZ_t$, $\sum_{0<s\leq t}(\pi_s - \pi_{s^-}) = \int_0^t (\pi_s - \pi_{s^-})\,dN(s,z)$ with $N(s,z)$ a counting process with intensity $\lambda(X_t)$, and

$$E\left(\int_0^t \Psi_{s^-}\left(e^{\beta^\intercal(s)z}-1\right)dN(s,z)\right) = \int_0^t \Psi_{s^-}\lambda(X_{s^-})\left(M_z(\beta(s))-1\right)ds$$

Now, it is easy to see that the second integral is a (complex-valued) martingale as well as the compensated jump process given by the last two integrals. Because $\Psi_t^*$ is itself a martingale, its drift has to be zero. Use assumption (1) to replace $\mu(X)$, $\sigma(X)$, and $\lambda(X)$ with their expressions, we end-up with the ODE's in (6). Conditions (1) and (2 ) are necessary for a stochastic integral to be a martingale, while condition (3) is a basic martingale requirement. $\square$

The guess for the conditional expectation in (6) makes it easier to compute the Fourier transform of the price. Denote by $G_{a,b}(y, X_0, T)$ the price at $X_0$ of a security that pays $\exp(a^\intercal X_T)$ at time $T$ when $b^\intercal X_T \leq y$, where $y$ is a contract parameter and $a$, $b$ are vectors of constants, and denote its Fourier transform by $\widehat{G}_{a,b}(y, X_0, T)$. From Fourier theory we have that

$$\widehat{G}_{a,b}(v, X_0, T) = \int_{\mathbb{R}} e^{ivy}\,dG_{a,b}(y, X_0, T)$$

The Fourier transform is

$$\begin{aligned}\widehat{G}_{a,b}(v, X_0, T) &= \int_{\Omega} \delta_T e^{(a+ivb)\cdot X_T}\,dQ(\omega) \\ &= \Psi(a+ivb, X_0, 0, T)\end{aligned} \tag{7}$$

where we have used in the first equality the fact that $G_{a,b}(y, X_0, T)$ is positive and increasing.

This result is very important, since the Fourier transform of the price is easier to obtain than the price itself. The obvious reason is that the payoff in (3) is not a smooth function (not differentiable everywhere) due to the presence of indicator functions, while the Fourier transform of the price is a smooth function of the state vector.

So far, only part of the the general payoff function in (3) has been considered. Some pricing problems involve payoffs of the form $v^\intercal X\exp(u^\intercal X)$ such as Asian options or options on bond yields. Below we show how the Fourier transform methodology can handle this type of contingent claims as well. Setting

$$\xi(v,u,x,t,T) = E^Q\left[\left.\frac{\delta_T}{\delta_t}v^\intercal X_T\exp(u^\intercal X_T)\right|\mathcal{F}_t\right] \tag{8}$$

as the price of this payoff, it is enough to note that

$$\begin{aligned}\xi(v,u,x,t,T) &= E^Q\left[\left.\frac{\delta_T}{\delta_t}v^\intercal\nabla_u\left(\exp(u^\intercal X_T)\right)\right|\mathcal{F}_t\right] \\ &= v^\intercal\nabla_u\Psi(u,x,t,T)\end{aligned}$$

if it is safe to differentiate under the integral sign, where $\nabla_u(\exp(u^\intercal X_T))$ is the gradient of $\exp(u^\intercal X_T)$ with respect to $u$. Under the affine assumption, we can show that

$$\xi(v,u,x,t,T) = \Psi(u,x,t,T)\left(A(t) + B(t)^\intercal x\right) \tag{9}$$

subject to solving for the set of ODE's:

$$-\dot{B}(t) = k_1^\intercal B(t) + \beta(t)^\intercal H_1 B(t) + l_1\nabla M_z(\beta(t))B(t) \tag{10}$$

$$-\dot{A}(t) = k_0 \cdot B(t) + \beta(t)^\intercal H_0 B(t) + l_0 \nabla M_z(\beta(t))^\intercal B(t)$$

with boundary conditions $B(T) = \upsilon$, and $A(T) = 0$.

**Proposition 2** *Suppose that:*

1. $E\left(\int_0^T |\widetilde{\gamma}_t| \, dt\right) < \infty$ *where* $\widetilde{\gamma}_t = \left(\xi_t \left(M_z\left(\beta(t)\right) - 1\right) + \Psi_t\left(M_z\left(\beta(t)\right) - 1\right)\right) \lambda\left(X_t\right)$*;*
2. $E\left(\left(\int_0^T \widetilde{\eta}_t \cdot \widetilde{\eta}_t dt\right)^{\frac{1}{2}}\right) < \infty$ *where* $\widetilde{\eta}_t = \left(\xi_t B(t)^\intercal + \Psi_t \beta(t)^\intercal\right) \sigma\left(X_t\right)$*; and*
3. $E\left(\left|\left(-\int_0^T r\left(X_s\right) ds\right) \xi_T\right|\right) < \infty,$

*then the solution to (8) is given by (9).*

**Proof.** See Appendix A. □

Let $H^c_{a,b}(y, X_0, T)$ be the price at $X_0$ of a security that pays $c^\intercal X_T e^{a^\intercal X_T}$ when $b^\intercal X_T \leq y$. The Fourier transform of the price is

$$\begin{aligned} \widehat{H}^c_{a,b}\left(\nu, X_0, T\right) &= \int_{\mathbb{R}} e^{i\nu y} dH^c_{a,b}(y, X_0, T) \\ &= \xi\left(c, a + i\nu b, X_0, 0, T\right) \end{aligned}$$

by following the same steps as in equation (7).

After computing the Fourier transform, one is interested in going backwards and recover the original price. Under the integrability condition $\int_{\mathbb{R}} |\Psi\left(a + ivb, X_0, 0, T\right)| \, d\nu < \infty$, it can be shown that (see Duffie et al. Appendix A for a proof) $G_{a,b}\left(y, X_0, T\right)$ is given by

$$G_{a,b}\left(y, X_0, T\right) = \frac{\Psi\left(a, X_0, 0, T\right)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im\left[\Psi\left(a + i\nu b, x, t, T\right) e^{-i\nu y}\right]}{\nu} d\nu \tag{11}$$

where $\Im\left[\cdot\right]$ stands for the imaginary part of a complex number. In the same way, the function $H^c_{a,b}(y, X_0, T)$ is recovered from

$$H^c_{a,b}(y, X_0, T) = \frac{\xi\left(c, a, X_0, 0, T\right)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im\left[\xi\left(c, a + i\nu b, x, t, T\right) e^{-i\nu y}\right]}{\nu} d\nu \tag{12}$$

subject to $\int_{\mathbb{R}} |\xi\left(c, a + ivb, X_0, 0, T\right)| \, d\nu < \infty$.

In practice, solving the integrals in the Fourier inversion can be cumbersome. One may rely on numerical integration techniques such as quadratures or fast inversion algorithms. Since it is known from Fourier Inversion theorem that

$$G_{a,b}\left(y, X_0, T\right) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi\left(a + i\nu b, x, t, T\right) e^{-i\nu y} d\nu$$

and

$$H^c_{a,b}(y, X_0, T) = \frac{1}{2\pi} \int_{\mathbb{R}} \xi\left(c, a + i\nu b, x, t, T\right) e^{-i\nu y} d\nu$$

one can take advantage of the efficient Fast Fourier Transform algorithm to compute the two integrals above. Whether the payoff be on the level or the exponential of the uncertainty, one can prove that analyticity of the characteristic function is sufficient to obtain the Fourier transform of many prices in closed form. This is the focus of the next paragraph.

## 2.2 The Bakshi and Madan approach

The approach introduced in Bakshi and Madan [8] is more general in that it is not constrained to a specific probablistic context. Note that equation (4) defines the characteristic function of the state price density (the Bond price times the Radon-Nikodym derivative) for $u = i\varphi$. The authors showed that this is a sufficient statistic to handle many pricing problems, even if the payoff has the form in (2). Indeed, $\exp(i\varphi^\intercal X) = \cos(\varphi^\intercal X) + i \sin(\varphi^\intercal X)$ by Euler's Formula, so that the valuation equation (4) will return the price of two hypothetical securities that pay cosine and sine of the state vector. Bakshi and Madan proved that every payoff in $L^1$ is in the span of these two securities in that there exists a continuum of coefficients $w(\varphi) \in L^1$ such that the payoff can be written as

$$\chi(X) = \int_{\mathbb{R}^k} \Re\left[w(\varphi)^\top \exp(i\varphi \cdot X)\right] d\varphi \tag{13}$$

where $\Re[\cdot]$ denotes the real part of a complex number. The coefficients $w(\varphi)$ represent a (complex valued) trading strategy that tells how much to hold in the sine and cosine securities in order to replicate the $L^1$ payoff. It is not hard to guess the formula for $w(\varphi)$. Indeed, from standard Fourier theory

$$w(\varphi) = w_1(\varphi) + iw_2(\varphi) = \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \chi(X) e^{-i\varphi \cdot X} dX \tag{14}$$

which is called the inverse Fourier transform of $\chi(X)$. From the last equation we clearly see that

$$\begin{aligned} w_1(\varphi) &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \chi(X) \cos(\varphi \cdot X) dX \\ w_2(\varphi) &= -\frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \chi(X) \sin(\varphi \cdot X) dX \end{aligned}$$

The price of this trading strategy is $\int_{\mathbb{R}^k} \Re[\Psi(i\varphi, X, t, T) w(\varphi)] d\varphi$. Substituting for $w(\varphi)$ from equation (14) we have

$$p_t = \frac{1}{(2\pi)^{k\times k}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \Re\left[\Psi(i\varphi, X, t, T) \chi(X) e^{-i\varphi \cdot X}\right] dX d\varphi$$

which is the price of the contingent claim $\chi(X)$, otherwise arbitrage opportunities would arise. For a put option we have $\chi(X) = \max(0, K - X)$. This implies that $w_1(\varphi) = (1 - \cos(\varphi K)) / 2\pi\varphi^2$ and $w_2(\varphi) = -\left(K/\varphi - \sin(\varphi K)/\varphi^2\right)/2\pi$. For a general payoff such as in equation (3), the $L^1$ condition may very well be violated. It is possible then to adjust the payoff by an affine function $\gamma_0 + \gamma_1^\intercal X$, for $\gamma_0$ constant and $\gamma_1 \in \mathbb{R}^k$, such that $\chi(X) + \gamma_0 + \gamma_1^\intercal X$ equals the right hand side in (13). For instance, a call option payoff verifies the identity $\max(0, X - K) = \max(0, K - X) + X - K$ whence $\gamma_0 = -K$ and $\gamma_1 = 1$.

To show that equation (4) is a powerful pricing engine, it can be differentiated as many times as needed, thus recovering the prices of polynomial payoffs. Assuming that the $n$-th moment of $X$ (with respect to $\mathbf{Q}$) exists, it is true that

$$\left| \int_\Omega \exp\left(-\int_t^T r(X_s) ds\right) (iX)^n \exp(iu^\intercal X) dQ(\omega) \right| < \infty$$

which reminds us of the gradient depicted in formula (12) with $n = 1$. At this level it is important to note that for $n \geq 1$, polynomial payoffs are not in $L^1$. The direct spanning via characteristic functions will be feasible if $\int_\Omega |X|^n dQ(\omega) < \infty$ and the payoff is restricted to some compact set in $\mathbb{R}^k$, otherwise one should adjust the spanning by some polynomial as in the call option case. It turns out that differentiation is not the only way to recover prices from (4), translation is also a feasible approach, especially when the payoff is on the exponential of $X$. This will collapse to the same original equation (4) with perhaps a different parameter $\varphi$. Hence, besides the fact that the Bakshi and Madan [8] approach is a generalization of the results in Duffie et al. [2], it has a very elegant economic side as well. The examples below should help further understand this approach.

# 3 Multidimensional payoffs

Multidimensional payoffs are hard to synthesize in a compact formula as in (3). We can, however, adopt the following notation

$$\chi^n(X) \equiv F\left(f_1\left(X\right), f_2\left(X\right), \cdots, f_n\left(X\right)\right)$$

for some functions $F$, $f_1$, $f_2$, ..., $f_n$. Taking the case $n = 2$, one example is

$$\chi^2(X) = \left(e^{u_1^\top X} - e^{u_2^\top X}\right)\mathbf{1}_{\left\{w_1^\intercal X \in \Lambda_1\right\}\cap\left\{w_2^\intercal X \in \Lambda_2\right\}}$$

which is typically the payoff of a correlation option. The notation is then similar to that in equation (3) except that the dimension is expanded. For illustrative purposes, we will show, using the example above, that even in a multidimensional setting, the analyticity of the Fourier transform is guaranteed. Set $\Lambda_1 = (-\infty, y_1], \Lambda_2 = (-\infty, y_2]$, the price of the contingent claim is

$$U\left(y_1, y_2\right) = E^Q\left[\delta_T\left(\exp\left(u_1^\top X\right) - \exp\left(u_2^\top X\right)\right)\mathbf{1}_{\left\{w_1^\intercal X \le y_1\right\}\cap\left\{w_2^\intercal X \le y_2\right\}}\right].$$

Let $\widehat{U}\left(v_1, v_2\right)$ be the Fourier transform of $U\left(y_1, y_2\right)$. We have

$$\begin{aligned}
\widehat{U}\left(v_1, v_2\right) &= \int_{\mathbb{R}^2} e^{i(v_1 y_1 + v_2 y_2)} dU\left(y_1, y_2\right) \\
&= \int_\Omega \delta_T e^{(u_1 + iv_1 w_1 + iv_2 w_2)^\top X_T} dQ\left(\omega\right) - \\
&\quad \int_\Omega \delta_T e^{(u_2 + iv_1 w_1 + iv_2 w_2)^\top X_T} dQ\left(\omega\right) \\
&= \Psi\left(u_1 + iv_1 w_1 + iv_2 w_2, X_0, 0, T\right) - \Psi\left(u_2 + iv_1 w_1 + iv_2 w_2, X_0, 0, T\right)
\end{aligned}$$

where the function $\Psi\left(\cdot\right)$ is the same as in the preceding section. The third line above is true if both $\Psi\left(u_1 + iv_1 w_1 + iv_2 w_2, X_0, 0, T\right)$ and $\Psi\left(u_2 + iv_1 w_1 + iv_2 w_2, X_0, 0, T\right)$ are finite. For the second equality, it is obtained by Fubini's theorem.

There is a special case when the contingency region $\mathbf{1}_{\left\{w_1^\intercal X \in \Lambda_1\right\}\cap\left\{w_2^\intercal X \in \Lambda_2\right\}}$ is stochastic, that is, one of the boundaries of the Borel sets is defined from a random variable. This case was discussed in Dempster and Hong [4] with a focus on a spread option. The authors circumvented the stochastic boundary by approximating it with rectangular areas. However, this could at most give them an upper and lower bound for the option price, hoping that as the discretization mesh tends to zero, the two bounds converge simultaneously to the true option price.

Inverting the Fourier transform does not carry much difference compared to equation (11) except that the inversion formula will now involve a two- ($n-$ for the general case) dimensional integral. Hence, the only difficulty in this setting is the computation of these higher dimensional integrals. However, the FFT methodology is still applicable, with a greater gain in accuracy and speed (see Dempster and Hong [4]).

We shall now lower the level of abstraction and see how the Fourier transform methodology yields the prices of specific contingent claims.

# 4 Examples of contingent claims

As in Duffie et al., in this section we assume that the underlying asset price processes $S_t$ under the probability measure $Q$ is related to the state vector by the functional form[3]

$$S_t = \left(\overline{a}(t) + \overline{b}(t)^\intercal X_t\right) e^{\widehat{a}(t) + \widehat{b}(t)^\intercal X_t}$$

[3]Duffie et al. considered two cases for the price process. Either start directly by specifying its behavior under the probability measure $Q$, or start by specifying the price dynamics under the original measure $P$. In the latter case we should define the process of the state price density (assumed to be exponential affine in Duffie et al.) and find the right dynamics of the price under the measure $Q$. This approach will add few irrelevant steps to our analysis, so we opted for the former that is $X_t$ is AJD under $Q$.

for some deterministic coefficients $\overline{a}(t)$, $\overline{b}(t)$, $\widehat{a}(t)$, and $\widehat{b}(t)$. Note that we could let the log-price be one of the elements of the state vector $X$. This is less straightforward (but easier) than the previous assumption, knowing that an asset price is a contingent claim on the dividends it pays. Throughout we will maintain the assumption that $X_t$ is the log-price meaning that $\overline{a}(t) = 1$, $\overline{b}(t) = \widehat{a}(t) = 0$, $\widehat{b}(t) = e(j)$, where $e(j)$ is a $k$ vector with 1 for the $j$-th component and 0 for the rest.

### 4.1 Call and put options

We now show in details how to apply the results of the previous section to these two contingent claims in detail. For the next contingent claims, we will skip many of the steps that will become redundant after this example. Recalling equation (3), a plain vanilla call option with strike $K$ that expires at time $T$ has payoff

$$\begin{aligned}\chi(X_T) &= \left(\exp\left(e\left(j\right)^\intercal X_T\right) - K\right)\mathbf{1}_{\{e(j)^\intercal X_T \geq \ln(K)\}} \\ &= \exp\left(e\left(j\right)^\intercal X_T\right)\mathbf{1}_{\{-e(j)^\intercal X_T \leq -\ln(K)\}} - K\exp\left(0^\intercal X_T\right)\mathbf{1}_{\{-e(j)^\intercal X_T \leq -\ln(K)\}}.\end{aligned} \tag{15}$$

Using our previous results, it easy to note that

$$p_0^C = G_{e(j),-e(j)}\left(-\ln(K), X_0, T\right) - KG_{0,-e(j)}\left(-\ln(K), X_0, T\right). \tag{16}$$

A put option (with the same maturity and strike) can be handled either directly in the same way or by put-call parity yielding the put option price

$$p_0^P = KG_{0,e(j)}\left(\ln(K), X_0, T\right) - G_{e(j),e(j)}\left(\ln(K), X_0, T\right). \tag{17}$$

Applying the Bakshi and Madan [8] approach, we recover the call price from the Fourier transform of the state price density. In this setting, the call option price has four elements; two prices and two probabilities (Arrow-Debreu prices). We denote the Fourier transform by $f\left(u,t,T\right)$. From equation (4), the call price is

$$\begin{aligned}p_t^C &= f\left(-ie(j),t,T\right)\frac{\int_\Omega \frac{\delta_T}{\delta_t} e^{e(j)^\intercal X_T}\mathbf{1}_{\{e(j)^\intercal X_T \geq \ln(K)\}} dQ\left(\omega\right)}{f\left(-ie(j),t,T\right)} \\ &\quad -Kf\left(0,t,T\right)\frac{\int_\Omega \frac{\delta_T}{\delta_t}\mathbf{1}_{\{e(j)^\intercal X_T \geq \ln(K)\}} dQ\left(\omega\right)}{f\left(0,t,T\right)}.\end{aligned} \tag{18}$$

Relying on the strict positivity of the terms $f\left(-ie(j),t,T\right)$[4] and $f\left(0,t,T\right)$, we can consider each of the terms $\frac{\delta_T e^{e(j)^\intercal X_T}}{\delta_t f(-ie(j),t,T)}$ and $\frac{\delta_T}{\delta_t f(0,t,T)}$ as Radon-Nikodym derivatives corresponding to another probability measure. Thus, the integrals above will return a well behaved probability $\left(\Pr\left(X_T > \ln(K)\right)\right)$ lying between 0 and 1. Denote these two probabilities as $\Pi_1\left(t,T\right)$ and $\Pi_2\left(t,T\right)$ and their corresponding characteristic functions by $f_1\left(u,t,T\right)$ and $f_2\left(u,t,T\right)$, then it is true that

$$\begin{aligned}f_1\left(u,0,T\right) &= \frac{f\left(u-e(j),0,T\right)}{f\left(-ie(j),0,T\right)} \\ f_2\left(u,t,T\right) &= \frac{f\left(u,0,T\right)}{f\left(0,0,T\right)}\end{aligned}$$

and from the results in Gil-Pelaez [5] on the inversion of characteristic functions, we have[5]

$$\Pi_1\left(0,T\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iu_j\ln(K)}f_1\left(u,0,T\right)}{iu_j}\right]du_j \tag{19}$$

where $u_j$ is the $j$-th position in the vector $u$ corresponding to the asset price logarithm. A similar formula holds for $\Pi_2\left(0,T\right)$ except that we replace $f_1\left(u,0,T\right)$ by $f_2\left(u,0,T\right)$. Gathering the terms that constitute the call price, we obtain

$$p_0^C = f\left(-ie(j),0,T\right)\Pi_1\left(0,T\right) - Kf\left(0,0,T\right)\Pi_2\left(0,T\right). \tag{20}$$

[4] From the properties of the probability measure $Q$, this term corresponds to the spot asset price at time $t$.

[5] Here, the option payoff depends only on one underlying asset. For a multidimentional payoff, the integral dimension should increase accordingly.

Equation (20) was the first to appear in the literature applying Fourier Transform techniques. However, the Bakshi and Madan [8] approach distinguishes itself from the previous literature in that they recover every contingent claim price from a single characteristic function (equation (4)), whereas the previous literature used at least two of them. For instance, in the call option case both $f_1(u,t,T)$ and $f_2(u,t,T)$ are translates of each other (after adjusting for the constants), so solving for one will lead to the other.

Comparing equation (11) and equation (19), we see that the latter is a scaled version of the former where the scale is either $f(-ie(j),0,T)$ (compare to $G_{e(j),0}(0,X_0,T)$ in (17)) or $f(0,0,T)$ (compare to $G_{0,0}(0,X_0,T)$ in (17)). To be more specific, the following equalities hold:

$$\begin{aligned}\Pi_1(0,T) &= \frac{G_{e(j),-e(j)}(-\ln(K),X_0,T)}{G_{e(j),0}(0,X_0,T)}\\ \Pi_2(0,T) &= \frac{G_{0,-e(j)}(-\ln(K),X_0,T)}{G_{0,0}(0,X_0,T)}.\end{aligned}$$

This confirms that the Bakshi and Madan [8] approach encompasses that of Duffie et al. [2] as we noted before. The results are established in the same fashion for a put option.

### 4.2 Double Strike Option

We consider an option that pays the same as a call option with strike $K$ if the asset price at time $T$ is in the interval $[K_1,K_2]$, that is:

$$\chi(X_T) = \left(e^{e(j)^\intercal X_T} - K\right)\mathbf{1}_{\{\ln(K_1)\le e(j)^\intercal X_T \le \ln(K_2)\}}.$$

Such an option is is similar to a Butterfly spread. Note the equality

$$\mathbf{1} = \mathbf{1}_{\{\ln(K_1)\le e(j)^\intercal X_T\le\ln(K_2)\}} + \mathbf{1}_{\{e(j)^\intercal X_T < \ln(K_1)\}} + \mathbf{1}_{\{e(j)^\intercal X_T > \ln(K_2)\}}$$

and use it to rewrite the payoff as

$$\chi(X_T) = \left(e^{e(j)^\intercal X_T} - K\right) - \left(e^{e(j)^\intercal X_T} - K\right)\left(\mathbf{1}_{\{e(j)^\intercal X_T\le\ln(K_1)\}} + \mathbf{1}_{\{e(j)^\intercal X_T\ge\ln(K_2)\}}\right)$$

and the price as

$$\begin{aligned}p_0^D &= S_0 - KG_{0,0}(0,X_0,T) - G_{e(j),e(j)}(\ln(K_1),X_0,T) - G_{e(j),-e(j)}(-\ln(K_2),X_0,T) +\\ &\quad K\left(G_{0,e(j)}(\ln(K_1),X_0,T) + G_{0,-e(j)}(-\ln(K_2),X_0,T)\right).\end{aligned}$$

This contract is therefore a equivalent to a combination of three securities: a long position in a forward contract with settlement price $K$, a long position in a put option with two strikes ($K$ to determine the payoff, and $K_1$ to determine the exercise region) and a short position in a call option also with two strikes ($K$ to determine the payoff, and $K_2$ to determine the exercise region). Therefore this case is identical to the previous call and put option case up to adjusting for some constants.

### 4.3 Quanto Option

A Quanto option is a contract where the payoff is measured in one currency whereas the payment is done in another currency. Suppose that the asset underlying the option and the exchange rate are both elements of the state vector $X$ and that they occupy arbitrary $n$-th and $m$-th positions ($n,\ m\le k$) in that vector respectively. The payoff at time $T$ is $e^{e(m)^\intercal X_T}\left(e^{e(n)^\intercal X_T} - K\right)\mathbf{1}_{e(n)^\intercal X_T>\ln(K)}$. This payoff is very similar to that of a call option with one more vector coefficients $e(n)$ so the price is

$$p_0^Q = G_{e(n)+e(m),-e(n)}(-\ln(K),X_0,T) - KG_{e(m),-e(n)}(-\ln(K),X_0,T).$$

Under the Bakshi and Madan approach, the price of this claim is

$$\begin{aligned} p_0^Q &= f\left(-i\left(e(n)+e(m)\right),0,T\right)\frac{\int_\Omega \frac{\delta_T}{\delta_t}e^{(e(n)+e(m))^\intercal X_T}\mathbf{1}_{\{e(n)^\intercal X_T\geq \ln(K)\}}dQ\left(\omega\right)}{f\left(-i\left(e(n)+e(m)\right),0,T\right)} \\ &\quad -Kf\left(e(m),0,T\right)\frac{\int_\Omega \frac{\delta_T}{\delta_t}e^{e(m)^\intercal X_T}\mathbf{1}_{\{e(n)^\intercal X_T\geq \ln(K)\}}dQ\left(\omega\right)}{f\left(e(m),0,T\right)} \end{aligned}$$

and the two characteristic functions are defined analogously as in the call option

$$\begin{aligned} f_1\left(u,0,T\right) &= \frac{f\left(u-e(n)-e(m),0,T\right)}{f\left(-i\left(e(n)+e(m)\right),0,T\right)} \\ f_2\left(u,0,T\right) &= \frac{f\left(u-e(m),0,T\right)}{f\left(-ie(m),0,T\right)}. \end{aligned}$$

To obtain the probability $\Pr\left(e(n)^\intercal X_T > \ln(K)\right)$, one applies the inversion formula in (19) and the price formula is then similar to (20).

## 4.4 Option to exchange one asset for another

Given two asset prices $e^{e(n)^\intercal X_T}$ and $e^{e(m)^\intercal X_T}$, the payoff of this option at time $T$ is $\max(e^{e(n)^\intercal X_T}-e^{e(m)^\intercal X_T}, 0)$. The market value at time 0 is then

$$p_0^E = G_{e(n),e(m)-e(n)}\left(0,X_0,T\right) - G_{e(m),e(m)-e(n)}\left(0,X_0,T\right)$$

which is similar to a call option with strike price equal to the second asset price. The inversion formula for the two characteristic functions simplifies to

$$\Pi_l\left(0,T\right) = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \Re\left[\frac{f_l\left(u_n,0,T\right)}{iu_n}\right]du_n$$

with $l=1,2$. It is now straightforward to identify the characteristic functions:

$$\begin{aligned} f_1\left(u,0,T\right) &= \frac{f\left(u-e(n),0,T\right)}{f\left(-ie(n),0,T\right)} \\ f_2\left(u,0,T\right) &= \frac{f\left(u-e(m),0,T\right)}{f\left(-ie(m),0,T\right)}. \end{aligned}$$

## 4.5 Chooser option

This contract allows its holder to choose between two assets at time $T$. Assuming rationality of the holder, the payoff is $\max\left(\exp\left(e(n)^\intercal X_T\right),\exp\left(e(m)^\intercal X_T\right)\right)$ which can be rewritten as

$$\chi(X_T) = \exp\left(e(m)^\intercal X_T\right) + \max\left(\exp\left(e(n)^\intercal X_T\right) - \exp\left(e(m)^\intercal X_T\right),0\right).$$

This is equivalent to a long position in the second asset combined with another long position in an option to exchange the first asset for the second. As a result, the initial value of this claim is given by

$$p_0^H = G_{e(m),0}\left(0,X_0,T\right) + G_{e(n),e(m)-e(n)}\left(0,X_0,T\right) - G_{e(m),e(m)-e(n)}\left(0,X_0,T\right)$$

## 4.6 Asian options

This is an example of an option contract for which a closed form solution is rarely available. A newly issued Asian option (with continuous monitoring of the asset price) pays $\max\left(\frac{1}{T}\int_0^T e(n)^\intercal X_s ds - K, 0\right)$ at time $T$. Here $e(n)^\intercal X$ is no longer the log-price, but it is the price itself (or any other quantity being averaged). If some time $t$ has elapsed since the option issuance, then the payoff should be adjusted as follows

$$\chi(Y_{t,T}) = \max\left(\tfrac{1}{T}Y_{t,T} - \left(K - \tfrac{1}{T}Y_{0,t}\right),0\right)$$

where $Y_{t,T} = \int_t^T e(n)^\intercal X_s ds$. Because $Y_{0,t}$ is observed ($\mathcal{F}_t$ measurable) by time $t$, the option has a new strike price $K - \frac{1}{T}Y_{0,t}$. We can treat $Y_{t,T}$ as a new state variable and define $\widetilde{X} = (X, Y)$ a $k+1$ dimensional state vector. Other parameters stated in assumption (1) have to be adjusted accordingly. The short term interest rate and the jump parameter $\lambda$ should remain uncorrected with $Y_{t,T}$ so $\widetilde{\rho}_1 = (\rho_1, 0)$ and $\widetilde{l}_1 = (l_1, 0)$. Note also that $d\widetilde{X}_t = (dX_t, e(n)^\intercal X_t dt)$. The new component is therefore locally deterministic. Adjusting the other parameters in the assumption is straightforward.[6] Let $\widetilde{\delta}_t = \exp\left(-\int_0^t r\left(\widetilde{X}_s\right) ds\right)$, the option value at time $t$ is then

$$
\begin{aligned}
p_t^A &= \int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t}\left(\tfrac{1}{T}Y_{t,T} - \left(K - \tfrac{1}{T}Y_{0,t}\right)\right)\mathbf{1}_{\{Y_{t,T}\geq(TK-Y_{0,t})\}} dQ(\omega) \qquad (21)\\
&= \tfrac{1}{T}\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t} e(k+1)^\intercal \widetilde{X}_T \mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)\\
&\quad - \left(K - \tfrac{1}{T}Y_{0,t}\right)\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t}\mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)\\
&= \tfrac{1}{T}H^{e(k+1)}_{0,-e(k+1)}(Y_{0,t} - TK, \widetilde{X}_t, T) - \left(K - \tfrac{1}{T}Y_{0,t}\right)G_{0,-e(k+1)}\left(Y_{0,t} - TK, \widetilde{X}_t, T\right).
\end{aligned}
$$

The Bakshi and Madan [8] approach can again be applied. We suppose that $Y_{t,T}$ has been used in the computation of the characteristic function $f(u,t,T)$ as a new state variable, that is,

$$
f(u,t,T) = E^Q\left[\exp\left(-\int_t^T R(\widetilde{X}_s)ds + i\left(u_1X_1 + u_2X_2 + \cdots + u_{K+1}Y\right)\right)\middle|\, \mathcal{F}_t\right].
$$

Starting with equation (21)

$$
\begin{aligned}
p_t^A &= \tfrac{1}{T}\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t} e(k+1)^\intercal \widetilde{X}_T \mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)\\
&\quad - \left(K - \tfrac{1}{T}Y_{0,t}\right)\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t}\mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)\\
&= \frac{-if_{u_{k+1}}(0,t,T)}{T}\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t}\frac{e(k+1)^\intercal \widetilde{X}_T e^{0^\intercal \widetilde{X}_T}}{-if_{u_{k+1}}(0,t,T)}\mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)\\
&\quad - \left(K - \tfrac{1}{T}Y_{0,t}\right)f(0,t,T)\int_\Omega \frac{\widetilde{\delta}_T}{\widetilde{\delta}_t}\frac{e^{0^\intercal \widetilde{X}_T}}{f(0,t,T)}\mathbf{1}_{\left\{e(k+1)^\intercal \widetilde{X}_T\geq(TK-Y_{0,t})\right\}} dQ(\omega)
\end{aligned}
$$

where $f_{u_{k+1}}(0,t,T)$ is the derivative of $f(u,t,T)$ with respect to its $k+1$-th argument. The ratios inside the integrals can be considered as Radon-Nikodym derivatives corresponding to two probability measures. The probability $\Pr(Y_{t,T} \geq (TK - Y_{0,t}))$ can be recovered under these two measures. The two characteristic functions related to these probability measures are then

$$
\begin{aligned}
f_1(u,t,T) &= \frac{f_{k+1}(u,t,T)}{-if_{k+1}(0,t,T)}\\
f_2(u,t,T) &= \frac{f(u,t,T)}{f(0,t,T)}.
\end{aligned}
$$

The inversion formula is similar to (19) except that the strike price is now $\left(K - \frac{1}{T}Y_{0,t}\right)$.

### 4.7 Yield curve derivatives

A zero-coupon bond with maturity $T > 0$ is a claim that gives its holder one unit of numéraire in every state of nature, and is hence independent of the uncertainty $X$. The price is therefore $G_{0,0}(0, X_0, T)$. It is

[6] For instance, $\widetilde{K}_0 = (K_0, 0)$, $\widetilde{K}_1 = (K_1, e(n))$.

irrelevant to recover the price through the inversion formula (11) since the payoff is certain. Formula (5) yields the price as

$$\Lambda(0,T) = \exp\left(\alpha(0,T,0) + \beta(0,T,0)^{\intercal} X_0\right)$$

where $\Lambda(T,s)$ denotes the zero coupon Bond price at time $T$ with maturity $s$ and $\alpha(t,T,u)$, $\beta(t,T,u)$ are modified notations for the solution of the ODE's in (6). This shows that the yield is an affine function (subject to the regularity conditions in proposition (1)) of the uncertainty $X$ with deterministic coefficients. Generally speaking, without the affine assumption, the Bakshi and Madan [8] approach gives the price of a zero-coupon bond as $f(0,t,T)$ and hence the whole term structure is recoverable from a single formula.

A call option struck at $K$ on a zero-coupon bond that matures at time $s < T$ has payoff $\max(\Lambda(T,s) - K, 0)$. Specifically, at time $T$ we have

$$\begin{aligned}
\max\left(\Lambda(T,s) - K, 0\right) &= \max\left(\exp\left(\alpha(T,s,0) + \beta(T,s,0)^{\intercal} X_T\right) - K, 0\right) \\
&= e^{\alpha(T,s,0)}\left(\exp\left(\beta(T,s,0)^{\intercal} X_T\right) - e^{-\alpha(T,s,0)} K\right) \mathbf{1}_{\{\beta(T,s,0)^{\intercal} X_T \geq -\alpha(T,s,0) + \ln(K)\}}.
\end{aligned}$$

This payoff is similar to that of a Quanto option.

An interest rate cap is a very popular instrument in over-the-counter markets. It is a portfolio of caplets where each caplet pays $\tau \max\left(\mathcal{R}((l-1)\tau, l\tau) - \overline{r}, 0\right)$ at time $l\tau$, where $\mathcal{R}((l-1)\tau, l\tau)$ is the $\tau$-year floating interest rate as seen at time $(l-1)\tau$ and $\overline{r}$ is the capped rate. The tenor $\tau$ of the interest rate is the compounding frequency. By definition

$$\Lambda((l-1)\tau, l\tau) = \tfrac{1}{1+\tau\mathcal{R}((l-1)\tau, l\tau)}.$$

We can use this fact to price the caplet:

$$\begin{aligned}
Caplet(l) &= E^{Q}\left[\delta_{l\tau} \max\left(\tfrac{1}{\Lambda((l-1)\tau, l\tau)} - 1 - \tau\overline{r}, 0\right)\right] \\
&= E^{Q}\left[\delta_{(l-1)\tau} \frac{\delta_{l\tau}}{\delta_{(l-1)\tau}} \max\left(\tfrac{1}{\Lambda((l-1)\tau, l\tau)} - 1 - \tau\overline{r}, 0\right)\right] \\
&= E^{Q^*}\left[\delta_{(l-1)\tau} \max\left(1 - (1+\tau\overline{r})\Lambda((l-1)\tau, l\tau), 0\right)\right]
\end{aligned}$$

which is the price of a put option on a zero-coupon bond with face value $(1+\tau\overline{r})$ and strike price 1, where $Q^*$ is a probability measure with Radon-Nikodym derivative

$$\tfrac{dQ^*}{dQ} = \tfrac{\delta_{l\tau}/\delta_{(l-1)\tau}}{\Lambda((l-1)\tau, l\tau)}.$$

Our previous result applies to floorets as well. These are put options on the interest rate $\mathcal{R}((l-1)\tau, l\tau)$, so a flooret is a call option on the zero-coupon bond $\Lambda((l-1)\tau, l\tau)$ with the same face value and strike price as the caplet.

Sometimes interest rate derivatives are written on a particular yield of the term structure. These can provide a hedge against unwanted movements of interest rates, especially for government bond holders. Suppose that we have a put option with maturity $T$ on the $s$-maturity yield $\mathcal{Y}(T,s)$. The option price is then

$$\begin{aligned}
p_t^Y &= E^{Q}\left[\delta_T \max\left(\overline{\mathcal{Y}} - \alpha(T,s,0) + \beta(T,s,0)^{\intercal} X_T, 0\right)\right] \\
&= \left(\overline{\mathcal{Y}} - \alpha(T,s,0)\right) H^{0}_{0,-\beta(T,s,0)}\left(\overline{\mathcal{Y}} - \alpha(T,s,0), X_0, T\right) + H^{\beta(T,s,0)}_{0,-\beta(T,s,0)}\left(\overline{\mathcal{Y}} - \alpha(T,s,0), X_0, T\right)
\end{aligned}$$

where $\overline{\mathcal{Y}}$ is the yield strike.

### 4.8 Correlation option

A correlation option is a two-asset dependent payoff option with typical cash-flow at maturity $\max(e^{s_1} - K_1, 0) \times \max(e^{s_2} - K_2, 0)$. Because of the two-dimensional structure of this option, no closed-form solution is found outside the lognormal model. Suppose that log-prices $s_1$ and $s_2$ occupy arbitrary $n$-th and $m$-th position in the state vector $X$, then the option price can be split into four elements

$$\begin{aligned} p_t^R &= \int_\Omega \delta_T \left(e^{e(n)^\intercal X_T} - K_1\right)\left(e^{e(m)^\intercal X_T} - K_2\right) \mathbf{1}_{\left\{e(n)^\intercal X_T > \ln(K_1)\right\}} \mathbf{1}_{\left\{e(m)^\intercal X_T > \ln(K_2)\right\}} dQ\left(\omega\right) \\ &= \int_\Omega \delta_T e^{(e(n)+e(m))^\intercal X_T} \mathbf{1}_{\{e(n)^\intercal X_T > \ln(K_1)\}} \mathbf{1}_{\{e(m)^\intercal X_T > \ln(K_2)\}} dQ\left(\omega\right) - \\ & K_1 \int_\Omega \delta_T e^{e(n)^\intercal X_T} \mathbf{1}_{\{e(n)^\intercal X_T > \ln(K_1)\}} \mathbf{1}_{\{e(m)^\intercal X_T > \ln(K_2)\}} dQ\left(\omega\right) - \\ & K_2 \int_\Omega \delta_T e^{e(m)^\intercal X_T} \mathbf{1}_{\{e(n)^\intercal X_T > \ln(K_1)\}} \mathbf{1}_{\{e(m)^\intercal X_T > \ln(K_2)\}} dQ\left(\omega\right) + \\ & K_1 K_2 \int_\Omega \delta_T \mathbf{1}_{\{e(n)^\intercal X_T > \ln(K_1)\}} \mathbf{1}_{\{e(m)^\intercal X_T > \ln(K_2)\}} dQ\left(\omega\right). \end{aligned}$$

Dropping the indicator functions $\mathbf{1}_{\left\{e(n)^\intercal X_T > \ln(K_1)\right\}} \mathbf{1}_{\left\{e(m)^\intercal X_T > \ln(K_2)\right\}}$, each integral above is a special case of the characteristic function in (5). For example, the first integral corresponds to $f\left(-i\left(e(n)+e(m)\right), t, T\right)$.

Using the same methodology as in the simple call option, we identify the Radon-Nikodym derivative corresponding to another probability measure as $\frac{\delta_T \exp((e(n)+e(m))^\intercal X_T)}{f(-i(e(n)+e(m)),0,T)}$. The characteristic function associated with this measure is

$$f_1\left(u_n, u_m, 0, T\right) = \frac{f(v_{n,m},0,T)}{f(-i(e(n)+e(m)),0,T)}$$

where $v_{n,m}$ is a vector with $u_n - i$ and $u_m$ in its $n$-th and $m$-th entry respectively (both $u_n$ and $u_m$ are scalars) and 0 for every other component. The rest of the integrals can be treated similarly where each of them requires a new characteristic function.

It remains to compute the bivariate probability $\Pr\left\{(e(n)^\intercal X_T > \ln(K_1)) \cap (e(m)^\intercal X_T > \ln(K_2))\right\}$. The general result has been established by Shephard [9], who provided a formula for the inversion of a multidimensional characteristic function. Applying his Theorem (5) to our case, the joint probability is obtained as

$$\begin{aligned} \Pi_l\left(0,T\right) &= -\tfrac{1}{4} + \tfrac{1}{2}\Pr\left(e(n)^\intercal X_T > \ln(K_1)\right) + \tfrac{1}{2}\Pr\left(e(m)^\intercal X_T > \ln(K_2)\right) + \\ & -\tfrac{1}{(2\pi)^2}\int_0^\infty \int_0^\infty \left( \begin{array}{c} \Re\left[\frac{e^{-iu_n \ln(K_1) - iu_m \ln(K_2)} f_l(u_n,u_m,0,T)}{u_n u_m}\right] - \\ \Re\left[\frac{e^{-iu_n \ln(K_1) + iu_m \ln(K_2)} f_l(u_n,-u_m,0,T)}{u_n u_m}\right] \end{array} \right) du_n du_m. \end{aligned}$$

Note that the marginal probabilities $\Pr\left(e(n)^\intercal X_T > \ln(K_1)\right)$ and $\Pr\left(e(m)^\intercal X_T > \ln(K_2)\right)$ can be obtained from the (modified) characteristic functions $f_l\left(u_n, 0, t, T\right)$ and $f_l\left(0, u_m, t, T\right)$ respectively with a similar expression as in (19). The final (lengthy) formula for the option price is

$$\begin{aligned} p_t^R &= f\left(-i\left(e(n)+e(m)\right), 0, T\right)\Pi_1\left(0,T\right) - K_1 f\left(-ie(n), 0, T\right)\Pi_2\left(0,T\right) - \\ & K_2 f\left(-ie(m), 0, T\right)\Pi_3\left(0,T\right) - K_1 K_2 f\left(0,0,T\right)\Pi_4\left(0,T\right) \end{aligned}$$

The previous examples illustrated how to obtain prices by using their Fourier transforms or the characteristic function of the state price density. However, we need to solve for them in the first place. This is the object of the following section.

# 5 Examples of affine stochastic processes

The purpose here is to solve explicitly for the ODE's in either (6) or (10)[7] for $X$ in the affine class. Recall that all processes are defined under the equivalent measure $Q$.

[7] Recall that solving the ODE's in (6) is equivalent to solving for the characteristic function of the vector $X$.

### 5.1 The lognormal process

We first illustrate the Fourier transform methodology in the generic lognormal model of Black and Scholes. This is one of the few processes admitting closed-form solutions for plain vanilla contingent claims. We will see that for this model, not only analyticity of the solution to (7) is guaranteed, but the inverse transform as given in (18) is also analytic.

Under the lognormal model, the log-price process is

$$dX_t = \left(\overline{r} - \tfrac{\sigma^2}{2}\right) dt + \sigma dW_t.$$

Note that here $k = 1$, $k_0 = \overline{r} - \frac{\sigma^2}{2}$, $H_0 = 0$, and $k_1 = H_1 = 0$. The interest rate is set to a constant $\overline{r}$ while the jump component is absent. It is not difficult to solve for $\alpha(t)$ and $\beta(t)$ in this case. They are given by

$$\begin{aligned} \beta(t) &= u \\ \alpha(t) &= \left(-\overline{r} + \left(\overline{r} - \tfrac{\sigma^2}{2}\right) u + \tfrac{\sigma^2}{2} u^2\right) \tau \end{aligned}$$

where $\tau = T - t$. Replacing these solutions in equation (5), we have

$$\Psi\left(u, X_t, t, T\right) = \exp\left(\left(-\overline{r} + \left(\overline{r} - \tfrac{\sigma^2}{2}\right) u + \tfrac{\sigma^2}{2} u^2\right) \tau + X_t u\right). \tag{22}$$

In appendix B, we show that the price of a plain vanilla call option is

$$C_t = S_t N(b_1) - K \exp(-\overline{r}\tau) N(b_2) \tag{23}$$

which is exactly the famous Black and Scholes formula.

### 5.2 Square root volatility process

We now consider the square root volatility process first proposed by Heston [6],

$$dV_t = \kappa\left(\overline{V} - V_t\right) dt + \gamma\sqrt{V_t} dW_t.$$

Some derivatives are written on the average variance rate during a specified time interval $[t, T]$ like variance swaps or realized variance options. Pricing these contingent claims is a challenging task. Recently, Carr and Lee [10] proposed a valuation strategy based on traded plain vanilla instruments such as European options. Here, we propose a closed-form solution to the ODE's systems (6) and (10). Up to our knowledge, this is the first time a quasi-closed solution to such contingent claims is being proposed.

Specifically, denote by $\vartheta_{t,T} \equiv \int_t^T V_s ds$, the cumulative variance. The uncertainty dynamics become

$$d\begin{pmatrix} V_t \\ \vartheta_{t,T} \end{pmatrix} = \left[\begin{pmatrix} \kappa\overline{V} \\ 0 \end{pmatrix} + \begin{pmatrix} -\kappa \\ 1 \end{pmatrix} V_t\right] dt + \begin{pmatrix} \gamma\sqrt{V_t} \\ 0 \end{pmatrix} dW_t.$$

With the bivariate boundary condition $(\beta_1(T), \beta_2(T)) = (u_1, u_2)$, the solution to (6) is

$$\begin{aligned} \beta_1(\tau) &= \frac{\kappa}{\gamma^2} - \frac{a}{\gamma^2}\left[\frac{ce^{a\tau} - d}{ce^{a\tau} + d}\right] \\ \beta_2(\tau) &= u_2 \\ \alpha\left(\tau\right) &= -\overline{r}\tau + \frac{\kappa\overline{V}}{\gamma^2}\left[\left(a + \kappa\right)\tau + 2\ln\left(\frac{d + c}{ce^{a\tau} + d}\right)\right] \end{aligned}$$

where $a = \sqrt{\kappa^2 - 2u_2\gamma^2}$, $c = a - u_1\gamma^2 + \kappa$, $d = a - u_1\gamma^2 - \kappa$, $\overline{r}$ is the (assumed constant) interest rate and $\tau = T - t$. Working out the solution to $B_1(t)$ and $B_2(t)$ while considering the boundary conditions $(B_1(T), B_2(T)) = (v_1, v_2)$, we obtain

$$B_1(\tau) = \frac{\exp\left(a\tau\right)}{(c\exp(a\tau) + d)^2}\left(v_1 + \frac{v_2}{a}(d^2 - c^2)\right) - \frac{v_2}{\left(c\exp(a\tau) + d\right)^2}\left(\frac{d^2}{a} - \frac{c^2}{a}\exp(2a\tau) - 2dc\exp(a\tau)\tau\right)$$

$$B_2(\tau) = v_2$$

As for the solution of $A(t)$, we have

$$A(\tau) = \frac{\kappa\overline{V}}{(c\exp(a\tau)+d)a^2c}\left(v_2(c^2-d^2)+2ac^2\exp(a\tau)\tau - av_1\right) - \frac{\kappa\overline{V}}{a^2c(c+d)}\left(v_2(c^2-d^2)-av_1\right) - \frac{\kappa\overline{V}\tau}{a}v_2$$

given the boundary condition $A(T) = 0$.

Unfortunately, this volatility model is constrained in $u_2$ and the constraint is $u_2 \leq \frac{\kappa^2}{2\gamma^2}$. Nonetheless, this is enough to price options on the average variance since $u_2$ has to be set to 0 in that case. The very existence of the price itself is tantamount to the existence of the first moment of $\vartheta_{t,T}$. Performing the required algebra on the solution system, a closed-form expression for the expectation $E^Q\left[\vartheta_{t,t+\tau}\right]$ is easily obtainable. However, the result in equation (9) will return the discounted expectation. Given the assumption of constant interest rate, the discounting can be easily removed. Set $u = (0,0)^\intercal$, $v = (0,1)^\intercal$, and $X_t = (V_t, \vartheta_{t,t+\tau})$ the expectation is

$$\begin{aligned} E^Q\left[\vartheta_{t,t+\tau}\right] &= \exp(\overline{r}\tau)\xi\left(u,v,X_t,t,t+\tau\right) \\ &= \vartheta_{t,t} + \overline{V}\tau + \frac{\left(V_t - \overline{V}\right)}{\kappa}\left(1-\exp(-\kappa\tau)\right). \end{aligned}$$

A similar expression was derived by Dufresne [16] using a different methodology.

As a numerical example, we value a set of newly issued call and put options on the average variance under different parameter settings, and compare the Fourier Transform results with those from Monte Carlo simulation. This is the same exercise as pricing an Asian option, as outlined in the previous paragraph. The integrals in equations (11) and (12) were computed numerically using a Gauss-Lobatto quadrature by truncating the integration range to $[0.00001, 10000]$.

Table 1: Call option prices as returned by the Fourier transfrom mehod (FT) and Monte Carlo simulation (MC). The parameters are as follows: $V_0 = 0.0387$, $\vartheta_{0,0} = 0$, $\overline{r} = 0.1$ and $T = 0.25$. $k_v$ is the option's strike price. For the two columns FT1 and MC1, the square root parameters are $\overline{V} = 0.04$, $\kappa = 1.2$, $\gamma = 0.1$ and for FT2 and MC2 the square root parameters are $\overline{V} = 0.04$, $\kappa = 0.6$, $\gamma = 0.5$. Monte Carlo simulation was run using 1000 time steps and 10000 replications. Standard errors are in parantheses.

| $k_v$ | FT1 | MC1 | FT2 | MC2 |
|---|---|---|---|---|
| 0.01 | 0.02816393 | $0.02824543(4.95E-05)$ | 0.02841436 | $0.02804353(2.59E-04)$ |
| 0.02 | 0.01841081 | $0.01838643(4.94E-05)$ | 0.02022434 | $0.02021205(2.42E-04)$ |
| 0.03 | 0.00870894 | $0.00871673(4.92E-05)$ | 0.01419625 | $0.01435494(2.17E-04)$ |
| 0.04 | 0.00149922 | $0.00152676(2.72E-05)$ | 0.00993390 | $0.00950397(1.82E-04)$ |
| 0.05 | 0.00004425 | $0.00004524(4.43E-06)$ | 0.00650399 | $0.00669457(1.61E-04)$ |
| 0.06 | 0.00000029 | $0.00000002(1.52E-08)$ | 0.00397493 | $0.00422106(1.26E-04)$ |
| 0.07 | 0.00000025 | 0(0) | 0.00268307 | $0.00286076(1.06E-05)$ |

The results are given in Tables 1 and 2. The Fourier Transform price computation is very rapid as compared to Monte Carlo simulation; using MATLAB on a laptop with a 2.0Ghz processor the Fourier Transform price is computed in 0.109 seconds whereas the Monte Carlo price is returned in 2.125 seconds. Quadrature routines based on Gauss-Lobatto points are known to converge very rapidly, so the Fourier Transform price is very accurate as well. It is mainly for this reason that in some literature the Fourier Transform price is labeled as closed-form (see Heston [6] for example).

# 6 Conclusion

This paper reviewed the application of Fourier transform for option pricing. The methodology is basically a two-step process. The first step is to compute the Fourier transform of the price in closed-form. The

Table 2: Put option prices as returned by the Fourier transfrom mehod (FT) and Monte Carlo simulation (MC). The parameters are as follows: $V_0 = 0.0387$, $\vartheta_{0,0} = 0$, $\overline{r} = 0.1$ and $T = 0.25$. $k_v$ is the option's strike price. For the two columns FT1 and MC1, the square root parameters are $\overline{V} = 0.04$, $\kappa = 1.2$, $\gamma = 0.1$ and for FT2 and MC2 the square root parameters are $\overline{V} = 0.04$, $\kappa = 0.6$, $\gamma = 0.5$. Monte Carlo simulation was run using 1000 time steps and 10000 replications. Standard errors are in parantheses.

| $k_v$ | FT1 | MC1 | FT2 | MC2 |
|---|---|---|---|---|
| 0.03 | 0.00005123 | $0.00004928(3.68E-06)$ | 0.00562055 | $0.00575073(7.89E-05)$ |
| 0.04 | 0.00259461 | $0.00254389(3.13E-05)$ | 0.01111130 | $0.01091265(1.17E-04)$ |
| 0.05 | 0.01089274 | $0.01094775(4.90E-05)$ | 0.01743448 | $0.01735359(1.49E-04)$ |
| 0.06 | 0.02060188 | $0.02065037(4.97E-05)$ | 0.02465852 | $0.02497297(1.79E-04)$ |
| 0.07 | 0.03035493 | $0.03033056(4.93E-05)$ | 0.03311976 | $0.03304475(2.00E-04)$ |
| 0.08 | 0.04010704 | $0.04010414(4.99E-05)$ | 0.04213922 | $0.04184965(2.20E-04)$ |
| 0.09 | 0.04985955 | $0.04981762(4.93E-05)$ | 0.05100420 | $0.05071641(2.34E-04)$ |
| 0.1 | 0.05961490 | $0.05966308(5.01E-05)$ | 0.06010674 | $0.06075122(2.38E-04)$ |

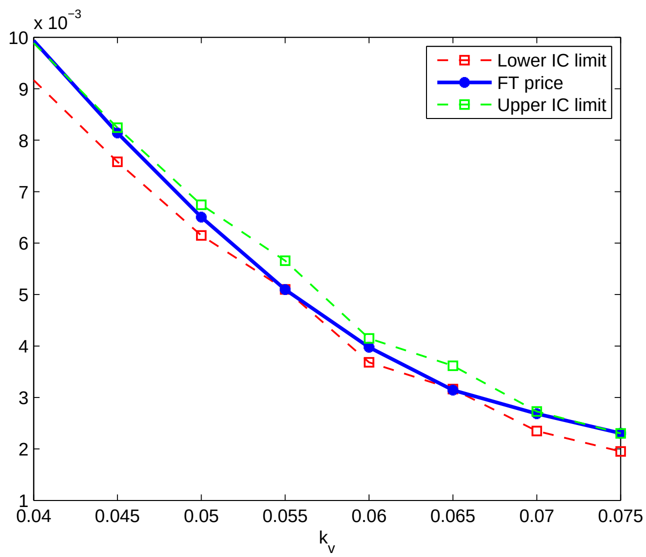


Figure 1: Average variance call price function. $k_v$ is the option's strike price. The interval bounds are at the 95% confidence level and generated using Monte Carlo simulation with 1000 time steps and 10000 replications.

second step is to invert it to recover the original price. For the first step, one of the main results in this respect is that of Duffie et al. [2]: If the state vector $X$ is an affine jump-diffusion under the pricing measure, one can nest the Fourier transform of many payoffs with similar structure into a single valuation equation. Specifically, prices of simple call and put options are the result of the same function modulo some simple algebraic manipulations. Duffie et al. [2] carried analyticity to the highest level with complicated payoffs that previous methodologies could not achieve. We cite in this respect Asian options which do not admit closed-form prices even under simple probability setups. Multidimensional payoffs can also be handled, and an example of a correlation option was given.

Relaxing any distributional assumptions of the vector $X$, a more powerful result was achieved by Bakshi and Madan. These authors retain the joint characteristic function of that vector as the only pricing engine. In many cases, manipulating the characteristic function through differentiation and/or translation lets the methodology encompass many contingent claims. Solving for this function is by far much easier that solving for the price directly, especially in the presence of jumps or stochastic volatility. Usually, an explicit solution for the Fourier transform of the price is available in closed-form up to solving a set of ODE's. The second

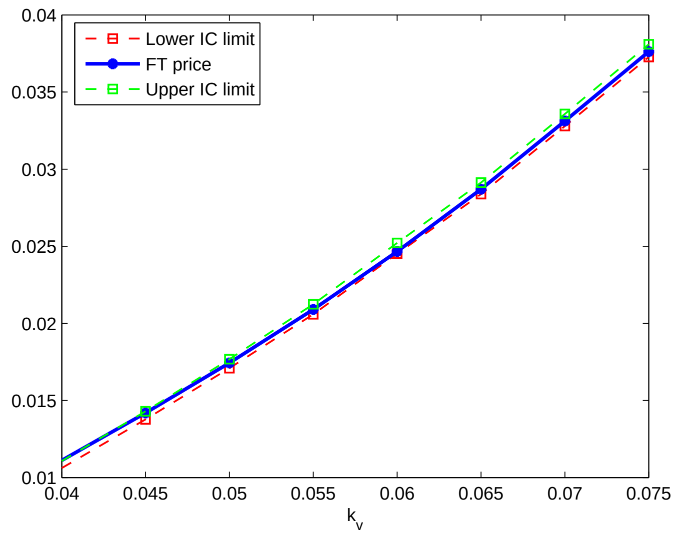


Figure 2: Average variance put price function. $k_v$ is the option's strike price. The interval bounds are at the 95% confidence level and generated using Monte Carlo simulation with 1000 time steps and 10000 replications.

step of the Fourier transform methodology corresponds to inverting the price transform. This task is (most of the time) achieved by numerical methods, and one can take advantage of the abundance of very efficient numerical techniques, such as the Fast Fourier Transform algorithm, to carry the inversion.

Within the realm of risk management, evaluating the sensitivity of option prices is as important as evaluating the price itself. Under some regularity conditions, option price sensitivities can be recovered from the Fourier transform of the price through a differentiation exercise. We believe that the approach can be extended in many directions. For example, how can it be applied to the evaluation of derivatives with an early exercise feature? To what extent does the methodology hold if $X$ is an arbitrary semimartingale process, rather than an affine jump diffusion? These questions will be addressed in a future research.

# A Proof of proposition 3

The proof follows in spirit that of proposition (1). Applying Itô's lemma to the discounted price $\xi^*(\upsilon, u, x, t, T) = \delta_t \xi(\upsilon, u, x, t, T)$, we have

$$\xi_t^* - \xi_0^* = \int_0^t -r\left(X_s\right) \xi_s^* ds + \int_0^t \exp\left(-\int_0^s r\left(X_u\right) du\right) d\xi_s. \tag{A.1}$$

As for $\xi_t$ itself, Itô's lemma implies

$$\begin{aligned}
\xi_t - \xi_0 = \int_0^t &\left( \begin{array}{c} \left(\dot{\alpha}\left(s\right) + \dot{\beta}\left(s\right)^\intercal X_{s-}\right) \xi_{s-} + \left(\dot{A}(s) + \dot{B}(s)^\intercal X_{s-}\right) \Psi_{s-} + \\ \left(\beta(s)^\intercal \xi_{s-} + B(s)^\intercal \Psi_{s-}\right) \mu\left(X_{s-}\right) \end{array} \right) ds \\
&+ \int_0^t \frac{1}{2} tr\left[\left(\beta\left(s\right)^\intercal \xi_{s-} \beta\left(s\right) + 2\beta\left(s\right)^\intercal B(s) \Psi_{s-}\right) \sigma\left(X_{s-}\right) \sigma\left(X_{s-}\right)^\intercal\right] ds \\
&+ \int_0^t \left(\beta\left(s\right)^\intercal \xi_{s-} + B(s)^\intercal \Psi_{s-}\right) \sigma\left(X_{s-}\right) dW_s + \int_0^t \left(\beta\left(s\right)^\intercal \xi_{s-} + B(s)^\intercal \Psi_{s-}\right) dZ_s \\
&+ \sum_{0<s\leq t} \left(\xi_s\left(X_s\right) - \xi_{s-}\left(X_{s-}\right) - \left(\beta\left(s\right)^\intercal \xi_{s-} + B(s)^\intercal \Psi_{s-}\right) \Delta X_s\right).
\end{aligned}$$

After introducing some simplifications, we can rewrite the above integrals as follows:

$$\xi_t - \xi_0 = \int_0^t \left( \begin{array}{c} \xi_{s^-}\left(\dot{\alpha}(s) + \dot{\beta}(s)^\intercal X_{s^-} + \beta(s)^\intercal \mu(X_{s^-})\right) ds \\ +\frac{1}{2} tr\left[\beta(s)^\intercal \sigma(X_{s^-})\sigma(X_{s^-})^\intercal \beta(s)\right] + \lambda(X_{s^-})(M_z(\beta(s)-1))\, ds \end{array} \right)$$
$$+ \int_0^t \xi_{s^-}\beta(s)^\intercal \sigma(X_{s^-})\, dW_s + \int_0^t \xi_{s^-}\left(e^{\beta(s)^\intercal z} - 1\right) dN(s,z)$$
$$- \int_0^t \xi_{s^-}\lambda(X_{s^-})(M_z(\beta(s)) - 1)\, ds$$
$$+ \int_0^t \left( \Psi_{s^-} \left( \begin{array}{c} \dot{A}(s) + \dot{B}(s)^\intercal X_{s^-} + B(s)^\intercal \mu(X_{s^-}) + tr\left[\beta(s)^\intercal B(s)\sigma(X_{s^-})\sigma(X_{s^-})^\intercal\right] \\ +\lambda(X_{s^-})\nabla M_z(\beta(s))^\intercal B(s) \end{array} \right) ds \right)$$
$$+ \int_0^t \Psi_{s^-} B(s)^\intercal \sigma(X_{s^-})\, dW_s + \int_0^t \Psi_{s^-} B(s)^\intercal z e^{\beta(s)^\intercal z} dN(s,z)$$
$$- \int_0^t \Psi_{s^-}\lambda(X_{s^-})\nabla M_z(\beta(s))^\intercal B(s) ds$$

where $E\left[z \cdot e^{\beta(s)\cdot z}\right] = \nabla M_z(\beta(t))$.[8] The first block of integrals (with $\xi_{s^-}$ in the integrand) is the same as in the proof of proposition (1) which will lead to the same ODE's in (6) after setting the drift to zero without omitting the first integral in (A.1). As for the second block, the drift has to be also zero (because $\Psi_t^*$ is a martingale from proposition (1)). Hence use assumption (1) to replace the coefficients with their expression, we necessarily arrive at the ODE's in (10).

# B Proof of Black and Scholes formula

Here we show how to recover the original Black and Scholes formula through the Fourier inversion technique. First, note that the price is given in equation (16). With $y$ denoting the log-strike price, we need to compute explicitly the two terms $G_{1,-1}(-y, X_t, T)$ and $G_{0,-1}(-y, X_t, T)$ using (12). Starting with the first term, we have

$$\begin{aligned} G_{1,-1}(-\ln(K), X_t, T) &= \frac{\Psi(1, X_t.t, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im\left[\Psi(1 - i\upsilon, X_t.t, T)\, e^{i\upsilon \ln(K)}\right]}{\upsilon} d\upsilon \\ &= \frac{\exp(X_t)}{2} + \frac{\exp(X_t)}{\pi}\int_0^\infty \frac{\exp\left(-\frac{1}{2}\sigma^2\tau\upsilon^2\right)\sin\left[\upsilon b_1 \sigma\sqrt{\tau}\right]}{\upsilon} d\upsilon \\ &= \exp(X_t)\left[\frac{1}{2} + \int_0^{b_1} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz\right] \\ &= \exp(X_t)\, N(b_1) \end{aligned}$$

where

$$\Im\left[\Psi(1 - i\upsilon, X_t, t, T)\, e^{i\upsilon \ln(K)}\right] = -\exp\left(X_t - \frac{1}{2}\sigma^2\tau\upsilon^2\right)\sin\left[\upsilon b_1 \sigma\sqrt{\tau}\right]$$

and

$$b_1 = \left(X_t - \ln(K) + \left(\overline{r} + \frac{\sigma^2}{2}\right)\tau\right)/\sigma\sqrt{\tau}.$$

The term $G_{0,-1}(-\ln(K), X_t, T)$ is computed with similar steps

$$\begin{aligned} G_{0,-1}(-\ln(K), X_t, T) &= \frac{\Psi(0, X_t, t, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im\left[\Psi(-i\upsilon, X_t.t, T)\, e^{i\upsilon \ln(K)}\right]}{\upsilon} d\upsilon \\ &= \frac{\exp(\overline{r}\tau)}{2} + \frac{\exp(\overline{r}\tau)}{\pi}\int_0^\infty \frac{\exp\left(-\frac{1}{2}\sigma^2\tau\upsilon^2\right)\sin\left[\upsilon b_2 \sigma\sqrt{\tau}\right]}{\upsilon} d\upsilon \end{aligned}$$

[8] The derivative of the expectation is the expectation of the derivative under some regularity conditions. See Rudin [3] (Theorem 42, page 236).

$$
\begin{aligned}
&= \exp\left(\overline{r}\tau\right)\left[\frac{1}{2}+\int_0^{b_2}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right] \\
&= \exp\left(\overline{r}\tau\right)N(b_2)
\end{aligned}
$$

with $b_2 = b_1 - \sigma\sqrt{\tau}$.

To compute the integral $\int_0^\infty \frac{\exp\left(-\frac{1}{2}\sigma^2\tau\upsilon^2\right)\sin\left[\upsilon b_1\sigma\sqrt{\tau}\right]}{\upsilon}d\upsilon$, we first introduce the notation $\alpha = \sigma\sqrt{\tau}$ and $\beta = b_2\sigma\sqrt{\tau}$. Then, we integrate by parts

$$
\int_0^\infty \exp\left(-\frac{1}{2}\alpha^2\upsilon^2\right)\frac{\sin\left[\upsilon\beta\right]}{\upsilon}d\upsilon = \alpha\int_0^\infty\int_0^\beta \frac{\sin\left[\upsilon z\right]}{z}\upsilon\exp\left(-\frac{1}{2}\alpha^2\upsilon^2\right)dzd\upsilon. \tag{24}
$$

Let $x = \sqrt{\alpha}\upsilon$, applying Fubini's theorem and performing necessary simplifications, the RHS of (24) becomes

$$
\int_0^{\frac{\beta}{\alpha}}\int_0^\infty \frac{\sin\left[xz\right]}{z}x\exp\left(-\frac{x^2}{2}\right)dxdz. \tag{25}
$$

Define $\phi\left(z\right) = \int_0^\infty \frac{\sin[xz]}{z}x\exp\left(-x^2\right)dx$. Following Rudin [3](Example 43, page 238), $\phi\left(z\right)$ is the solution to the ODE; $\frac{d\phi(z)}{dz} + z\phi\left(z\right) = 0$. The solution is trivial and $\phi\left(z\right) = \sqrt{\frac{\pi}{2}}\exp\left(-\frac{z^2}{2}\right)$ where the condition $\phi\left(0\right) = \sqrt{\frac{\pi}{2}}$ was used. Replacing these results into the price, we obtain equation (23).

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