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Quasi-maximum likelihood for estimating structural models

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Abstract : The estimation of the structural model poses a major challenge as its underlying asset (the firm's asset value) is not directly observable. We extend the maximum likelihood (ML) method of Duan (1994 and 2000), and propose a quasi-maximum likelihood (QML) approach that remains appropriate under alternative Markov assumptions, arbitrary debt payment schedules, and extended balance sheets. QML alternates between dynamic programming and maximum likelihood to simultaneously solve and estimate general structural settings. QML is highly flexible and effective. To support our construction, we conduct a numerical investigation and show that ML and QML agree in Merton's (1974) setting. Then, we achieve an empirical investigation, spotlight the credit-spread puzzle, and discuss a partial remedy via jumps and bankruptcy costs.

Keywords : Structural model, estimation, quasi-maximum likelihood, jump-diffusion processes, credit-spread puzzle

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1 Introduction

The aim of this paper is to estimate complex structural models under alternative underlying processes. Duan (1994 and 2000) developed a maximum likelihood (ML) approach to estimate Merton's (1974) model. Since then, the literature has reported several adaptations of ML used when corporate securities can be valued in closed form. We propose a quasi-maximum likelihood (QML) approach that remains appropriate under alternative underlying processes, arbitrary debt payment schedules, several seniority classes, and multiple intangible assets. QML is a highly flexible and effective method that alternates between dynamic programming (DP) and maximum likelihood (ML) to jointly solve and estimate structural settings.

The main difficulty in estimating structural models lies in the fact that their common state process (the firm's asset value) is not directly observable. The literature prior to Duan (1994 and 2000) reports several ad-hoc estimation procedures. The proxy approach combines market and accounting values of corporate securities as well as the historical and the implicit approaches to infer the model's unknown parameters (Brockman and Turtle 2003, Leland 2004, Bharath and Shumway 2008, Huang and Huang 2012, and Afik et al. 2016). The proxy approach suffers from a major drawback in that accounting values of corporate securities are not enough sensitive to changes in the firm's credit quality. The restricted volatility approach builds on equations that associate the non-observable firm's asset value and its volatility with the observable firm's equity value (market capitalization) and its volatility (Jones et al. 1984, Ronn and Verma 1986, Ogden 1987, Delianedis and Geske 2001 and 2003, Eom et al. 2004, Vassalou and Xing 2004, Hull et al. 2005, Chen et al. 2006, Charitou et al. 2013). Developed by Kealhofer, McQuown, and Vasicek, it gave rise to the well-known KMV licenced product, known now as MKMV after its acquisition by Moody's. KMV presents a shortcoming as it is specific to the lognormal assumption.

The ML approach of Duan (1994 and 2000) has given rise to several empirical investigations (Wang and Suo 2006, Li and Wong 2008, and Lee et al. 2015) and analytical modifications under 1– Merton's (1974) model (Duan et al. 2004 and 2005, Duan and Fulop 2009, Jovan and Ahcan 2017, Zhou and Zhang 2020, and Aguilar et al. 2021), 2– Leland's (1994) model (Ericsson and Reneby 2004 and 2005 and Forte and Lovreta 2012), 3– Brockman and Turtle's (2003) model (Wong and Choi 2009 and Dionne and Laajimi 2012), and 3– Black and Cox's (1976) model (Wong and Li 2006 and Amaya et al. 2019). The parallel between KMV and ML in Merton's (1974) setting is analysed by Duan et al. (2004) and Christoffersen et al. (2022).

When ML is unachievable, less demanding statistical principles are used, that is, 1– simulated ML (Bruche 2005), 2– GMM (Li et al. 2004, Hsu et al. 2010, and Huang et al. 2020), and 3– MCMC (Korteweg and Polson 2009 and Huang and Yu 2010).

QML combines between DP and ML to simultaneously solve and estimate structural settings. Since the quasi-likelihood expression cannot be computed in closed form, QML works as follows:

1. Derive the quasi-likelihood function;
2. Set a value for the vector of the model's unknown parameters;
3. Use dynamic programming and solve the model;
4. Extract the pseudo-time series of the firm's asset value in accordance with its associated time series of the firm's equity value (market capitalization);
5. Compute the value of the quasi-likelihood function;
6. Go to step 2 and repeat until the quasi-likelihoood function reaches a maximum (MADS algorithm is used herein).

QML is highly flexible and effective. To assess our construction, we use a Monte Carlo experiment in Merton's (1974) model, and show that QML and ML are consistent. Then, we conduct an empirical investigation of a speculative corporate debt portfolio. The credit spreads are under-valued under the pure diffusion lognormal assumption. We include tax benefits and bankruptcy costs, and experiment with the jump-diffusion Gaussian process. The credit spreads are adjusted high, which partially revisits the credit-spread puzzle.

The rest of this paper is organized as follows. Section 2 develops the QML approach, and Section 3 presents a case study. Section 4 concludes.

2 The QML approach

2.1 The model

We consider the extended structural setting of Ayadi et al. (2016), which accommodates alternative underlying processes, arbitrary interest and capital payment schedules, multiple seniority classes, and several intangible corporate securities. The firm's balance-sheet equality verifies

$$\begin{aligned} & a+\mathrm{TB}_t(a)-\mathrm{BC}_t(a) \\ = \quad & D_t^s(a)+D_t^j(a)+\mathcal{E}_t(a)\ , \\ & \text{almost surely for all } t \text{ and } a=A_t\ , \end{aligned} \tag{1}$$

where TB represents tax benefits, BC bankruptcy costs, D^s the senior debt, D^j the junior debt, $D^s+D^j=D$ the debt portfolio, $\mathcal{E}$ the firm's equity, and $a=A_t$ a given level of the firm's asset value at time t. The left-hand side of Equation (1), indicated by $\mathrm{TV}_t(a)$, is referred to as the total value of the firm at (t,a). The default event at a debt payment date t, defined as $\mathrm{TV}_t(a)\le D_t(a)$, is shown to take the form

$$a=A_t\le b_t,$$

where b_t is the endogenous default barrier at the debt payment date t. This structural model includes the fundamental settings of Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994). Ayadi et al. (2016) use dynamic programming (DP) under the lognormal assumption, and evaluate the corporate securities in Equation (1), seen as financial derivatives on the firm's asset value.

Along the same lines, Ben-Ameur et al. (2016) use DP under alternative jump-diffusion processes and evaluate Bermudan vanilla options. For the sake of clarity, we briefly present their dynamic program, as the same approach is herein used for valuing the corporate securities in Equation (1). Let X be the option underlying process, $\mathcal{P}=\{t_0,\dots,t_M\}$ the set of its exercise dates, where t_0 is the present date and t_M the maturity date, and $\mathcal{G}=\{a_1,\dots,a_p\}$ a mesh of grid points that spans the state space $(0,\infty)$ with the convention that $a_0=0$ and $a_{p+1}=\infty$. The option (known) exercise value function, holding value function, and overall value function at (t_m,x) are respectively indicated by $v_{t_m}^e(x)$, $v_{t_m}^h(x)$, and $v_{t_m}(x)$, where $x=X_{t_m}$ is a given level of the state variable X_{t_m} at time t_m. The authors start their backward numerical procedure at the option maturity, where the option overall value function is known in closed form, and propose an approximation $\widetilde{v}_{t_M}$ of v_{t_M} on $\mathcal{G}$ as follows:

$$\widetilde{v}_{t_M}(a_k)=v_{t_M}^e(a_k)\,,\quad \text{for } a_k\in\mathcal{G},$$

given $v_{t_M}^h$ is null at the option maturity. Then, they use a piecewise linear approximation $\widehat{v}_{t_M}$, and extend $\widetilde{v}_{t_M}$ from the grid points $\mathcal{G}$ to the overall state space $(0,\infty)$ as follows:

$$\widehat{v}_{t_M}(x)=\sum_{i=0}^{p}\left(\alpha_i^M+\beta_i^M x\right)\mathbb{I}\left(a_i<x\le a_{i+1}\right),\quad \text{for } x=X_{t_M}\in(0,\infty)\,,$$

where $\mathbb{I}$ is the indicator function. The local coefficients α_i^M and β_i^M of $\widehat{v}_{t_M}$ solve for

$$\widehat{v}_{t_M}(a_k)=\widetilde{v}_{t_M}(a_k)\,,\quad \text{for } a_k\in\mathcal{G},$$

while $\left(\alpha_0^M, \beta_0^M\right) = \left(\alpha_1^M, \beta_1^M\right)$ and $\left(\alpha_p^M, \beta_p^M\right) = \left(\alpha_{p-1}^M, \beta_{p-1}^M\right)$. Assume now that the model has been solved backward in time from the option maturity t_M to the debt payment date t_{m+1}, and that piecewise linear interpolations $\widehat{v}$ of v have been constructed on $(0,\infty)$ from t_M down to t_{m+1}. By no-arbitrage pricing, one has

$$v_{t_m}^h(x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\Delta t_m} v_{t_{m+1}}\left(X_{t_{m+1}}\right) \mid X_{t_m} = x\right],$$

where $\mathbb{Q}$ is a (the) risk-neutral probability measure, r the risk-free rate, and $\Delta t_m = t_{m+1} - t_m$. The expectation above cannot be computed in closed form and, thus, has to be approximated in some way. To that end, the authors exchange $v_{t_{m+1}}$ for

$$\widehat{v}_{t_{m+1}}(x) = \sum_{i=0}^{p}\left(\alpha_i^{m+1} + \beta_i^{m+1} x\right)\mathbb{I}\left(a_i < x \leq a_{i+1}\right), \quad \text{for } x \in (0,\infty),$$

and approximate v_{t_m} on $\mathcal{G}$ in quasi-closed form as follows:

$$\begin{aligned}\widetilde{v}_{t_m}^h(a_k) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-r\Delta t_m}\widehat{v}_{t_{m+1}}\left(X_{t_{m+1}}\right) \mid X_{t_m} = a_k\right] \\ =& e^{-r\Delta t_m}\sum_{i=0}^{p}\alpha_i^{m+1}T_{k,i}^0 + \beta_i^{m+1}T_{k,i}^1,\end{aligned}$$

where the α_i^{m+1} and β_i^{m+1} are the local coefficients of $\widehat{v}_{t_{m+1}}$, while the $T_{k,i}^0$ and the $T_{k,i}^1$ are the transition parameters of $X_{t_{m+1}}$ given that $X_{t_m} = a_k$, defined as follows:

$$\begin{aligned}T_{k,i}^0 =& \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}\left(a_i < X_{t_{n+1}} \leq a_{i+1}\right) \mid X_{t_m} = a_k\right] \\ =& \mathbb{Q}\left[X_{t_{n+1}} \in (a_i, a_{i+1}] \mid\mid X_{t_m} = a_k\right]\end{aligned}$$

and

$$T_{k,i}^1 = \mathbb{E}^{\mathbb{Q}}\left[X_{t_{n+1}}\mathbb{I}\left(a_i < X_{t_{n+1}} \leq a_{i+1}\right) \mid X_{t_m} = a_k\right].$$

Higher-order piecewise polynomials require higher-order transition tables. Piecewise linear polynomials achieve a good compromise between accuracy and computing time in addition to be simple to implement. For example, $T_{k,i}^0$ is the (risk-neutral) transition probability that the process X starts from a_k at t_m and visits the interval $(a_i, a_{i+1}]$ at t_{m+1}. An approximation $\widetilde{v}_{t_m}$ of v_{t_m} on $\mathcal{G}$ comes

$$\widetilde{v}_{t_m}(a_k) = \max\left(v_{t_m}^e(a_k), \widetilde{v}_{t_m}^h(a_k)\right), \quad \text{for } a_k \in \mathcal{G}.$$

Again, the authors use a piecewise linear interpolation $\widehat{v}_{t_m}$, and extend $\widetilde{v}_{t_m}$, defined on the grid points $\mathcal{G}$, to the overall state space $(0,\infty)$ as follows:

$$\widehat{v}_{t_m}(x) = \sum_{i=0}^{p}\left(\alpha_i^m + \beta_i^m x\right)\mathbb{I}\left(a_i < x \leq a_{i+1}\right), \quad \text{for } x \in (0,\infty),$$

The local coefficients α_i^m and β_i^m of $\widehat{v}_{t_m}$ solve for

$$\widehat{v}_{t_m}(a_k) = \widetilde{v}_{t_m}(a_k), \quad \text{for } x_k \in \mathcal{G},$$

while $(\alpha_0^m, \beta_0^m) = (\alpha_1^m, \beta_1^m)$ and $\left(\alpha_p^m, \beta_p^m\right) = \left(\alpha_{p-1}^m, \beta_{p-1}^m\right)$. The rest comes by backward induction.

Thus, for this dynamic program to run under alternative Markov processes, one mainly needs to derive and compute their associated transition parameters. See Ben-Ameur et al. (2016) for an explicit derivation of the transition parameters under jump-diffusion processes.

2.2 The likelihood function

Let θ be the vector of estimable parameters that characterise the Markov state process A (the firm's asset value), $f_{A_u,\theta}(. \mid a_t)$ the (known) conditional density function of A_u given $A_t = a_t$, for $u \geq t$, under the physical probability measure $\mathbb{P}$, and $\mathcal{P} = \left\{0, \ldots, t_0 = T, \ldots, t_M = T^D\right\}$ be the set of observation/payment/evaluation dates, where dates 0 and T are the first and last observation dates of the firm's equity value (market capitalization), while date T^D is the maturity date of the firm's debt portfolio. Date T also represents the present date.

Let $(e_0, \ldots, e_T)$ be the time series of the firm's market capitalization (the observations) and $(a_0, \ldots, a_T)$ be the time series of their associated firm's asset value (the pseudo observations). They verify

$$e = \mathcal{E}_t(a) \quad \text{or, equivalently,} \quad a = \mathcal{E}_t^{-1}(e), \tag{2}$$

for $e > 0$ and $a > b_t$, where b_t is the endogenous default barrier at $t \in \mathcal{P}$.

The likelihood function associated to the observed time series $(e_0, \ldots, e_T)$ of the Markov process $\mathcal{E}$ is

$$\begin{aligned}\mathcal{L}(\theta \mid e_0, \ldots, e_T) =& f_{\mathcal{E}_0,\theta}(e_0) \times \prod_{t=1}^{T} f_{\mathcal{E}_t,\theta}(e_t \mid e_{t-1}) \\ =& \prod_{t=1}^{T} f_{\mathcal{E}_t,\theta}(e_t \mid e_{t-1}),\end{aligned}$$

where $f_{\mathcal{E}_t,\theta}(. \mid e_{t-1})$ is the (unknown) conditional density function of $\mathcal{E}_t$ given $\mathcal{E}_{t-1} = e_{t-1}$ under the physical probability measure $\mathbb{P}$, with the convention that $f_{\mathcal{E}_0,\theta}(e_0) = 1$. We now rework this expression to depend on the (known) conditional density functions $f_{A_t,\theta}(. \mid a_{t-1})$.

For $e > 0$, the conditional cumulative density function of $\mathcal{E}_t$ given $\mathcal{E}_{t-1} = e_{t-1}$ verifies

$$\begin{aligned}F_{\mathcal{E}_t,\theta}(e \mid e_{t-1}) - F_{\mathcal{E}_t,\theta}(0 \mid e_{t-1}) =& \mathbb{P}(0 < \mathcal{E}_t \leq e \mid e_{t-1}) \\ =& \mathbb{P}\left(b_t < A_t \leq \mathcal{E}_t(e)^{-1} \mid a_{t-1}\right),\end{aligned}$$

given Equation (2). By differentiation with respect to $e > 0$, one has

$$\begin{aligned}f_{\mathcal{E}_t,\theta}(e \mid e_{t-1}) =& f_{A_t,\theta}\left(\mathcal{E}_t^{-1}(e) \mid a_{t-1}\right) \times \mathcal{E}_t^{-1}(e)' \\ =& \frac{f_{A_t,\theta}(a \mid a_{t-1})}{\mathcal{E}_t(a)'},\end{aligned}$$

for $e > 0$ and $a = \mathcal{E}_t^{-1}(e) > b_t$. This results in the likelihood function

$$\mathcal{L}(\theta \mid e_0, \ldots, e_T) = \prod_{t=1}^{T} \frac{f_{A_t,\theta}(a_t \mid a_{t-1})}{\mathcal{E}_t(a_t)'}, \tag{3}$$

where $a_t = \mathcal{E}_t^{-1}(e_t)$ is the pseudo-observation associated with the observation e_t, and $\mathcal{E}_t(a_t)'$ is the derivative of $\mathcal{E}_t$ evaluated at a_t, for $t \in \mathcal{P}$.

The likelihood expression in Equation (3) can be reworked otherwise if we consider the Markov underlying process $Z_t = \log(A_t)$. The same arguments give

$$\mathcal{L}(\theta \mid e_0, \ldots, e_T) = \prod_{t=1}^{T} \frac{f_{Z_t,\theta}(z_t \mid z_{t-1})}{a_t \times \mathcal{E}_t(a_t)'}, \tag{4}$$

where $z_0 = \log(a_0), \ldots, z_T = \log(a_T)$. The general expressions in Equation (3)–(4) hold for all structural settings, where the firm's asset value is the sole underlying process.

2.3 Resolution

The DP value functions of the corporate securities in Equation (1) are indicated by

$$\widehat{\mathrm{TB}},\ \widehat{\mathrm{BC}},\ \widehat{D},\text{ and }\widehat{\mathcal{E}}.$$

In particular, one has

$$\widehat{\mathcal{E}}_t\,(a) = \alpha_i^t + \beta_i^t a,$$

where $t \in \mathcal{P}$ and $a = A_t \in (a_i, a_{i+1}]$. For a fixed value of θ, we exchange the true value functions of corporate securities, default barriers, and the pseudo-observations by their known DP counterparts, and compute the quasi-likelihood function $\widehat{\mathcal{L}}$, while

$$\widehat{\mathcal{E}}_t\,(\widehat{a}_t)' = \beta_i^t, \quad \text{for } a_t \in (a_i, a_{i+1}]\,.$$

The rest consists of solving the likelihood equation

$$\widehat{\theta}^{\mathrm{QML}} = \arg\min_{\theta} -\log \widehat{\mathcal{L}}, \tag{5}$$

given the observed time series $e_0, \ldots, e_T$. This optimization problem is complex to achieve as $\widehat{\mathcal{L}}$ is not an explicit expression of θ. The grandient of $\widehat{\mathcal{L}}$ with respect to θ is unavailable. We use the mesh adaptive direct search (MADS) algorithm of Audet and Dennis (2006) to achieve the optimization problem in Equation (5). Figure 1 shows the shape of $\widehat{\mathcal{L}}$ under the lognormal assumption.

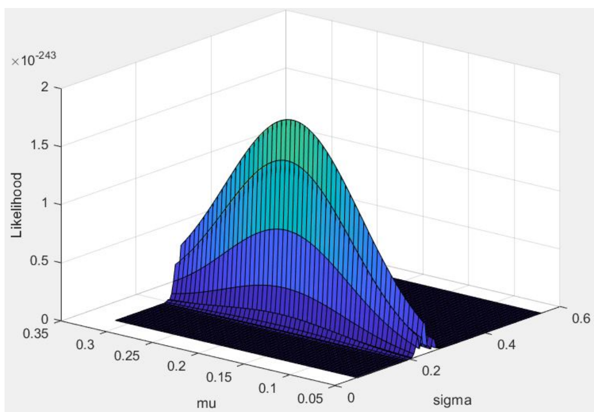

Figure 1: Curve of $\widehat{\mathcal{L}}$

3 Numerical and empirical investigation

The code lines are written in C and compiled under CPP. We use the GSL and NOMAD software libraries to ease solving the model and searching for the QML estimates. See Le Digabel (2011) for further details on NOMAD. To assess our QML approach, we propose a two-step approach.

The first experiment is based on a Monte Carlo simulation under Merton's (1974) setting. We set the model's true parameters as follows:

1. a drift and volatility of the underlying lognormal process $\mu_0 = 10\%$ (per year) and $\sigma_0 = 20\%$ (per year);
2. a risk-free rate $r = 5\%$ (per year);
3. a maturity and a principal amount of the pure corporate bond $T^D = 5$ years and $P = 100$ dollars;
4. a daily observation window $[0, T]$ of two years assuming 250 days per year.

We use the true value of $\theta_0 = (\mu_0, \sigma_0)$, simulate a daily path of A then of $\mathcal{E}$ over $[0, T]$ under the physical probability measure, estimate the model as if the simulated path of $\mathcal{E}$ were observed from the stock market, and provide the QML estimate of θ_0. Then, we perform $N = 2000$ replications of the same experiment, and compute the root mean square error (RMSE) of ML vs QML estimates of θ_0.

Table 1: RMSE of ML vs QML – Simulated Merton's (1974) setting

Estimation	Search method	RMSE($\widehat{\mu}$)	RMSE($\widehat{\sigma}$)
ML (Duan)	Newton-based	10.4%	4.5%
QML	Naive search	14.1%	5.1%
QML	MADS algorithm	12.9%	3.4%

The naive search is performed over a mesh of grid points of (μ, σ) of size 500^2. ML and QML estimates of θ_0 are comparable and show that the estimation of the volatility parameter is more accurate than of the drift parameter, as documented in the literature.

The second experiment is a case study of Phar-Mor, a public chain of discount drug stores. Phar-Mor went bankrupt in September 2001, while its debt maturity T^D was in March 2002. We set the present date T at one year then one year and a half before bankruptcy, and we observe the stock closing price over the 2-years time window $[0, T]$. The model respects the actual firm's interest and capital payment schedule over $\left[0, T^D\right]$. The risk-free rate is taken as the yield to maturity of a Treasury pure bond portfolio with an identical payment schedule to that of Phar-Mor.

Table 2 reports the QML estimates of θ in four structural settings à la Ayadi et al. (2016). The version with frictions includes tax benefits at the rate $r^c = 35\%$ and bankruptcy costs at the rate $w = 40\%$. Thus, under survival, the firm pays interest and saves taxes at the corporate tax rate $r^c = 35\%$, while, under liquidation, 40% of the remaining firm's asset value is used to finance bankruptcy costs. The rest is used to (partially) pay bondholders, while equityholders are not paid at all. We experiment with the pure diffusion lognormal process, where $\theta = (\mu, \sigma)$, vs the jump-diffusion Gaussian process, where $\theta = (\mu, \sigma, \lambda, \gamma, \delta)$. The parameters μ and σ are the drift and volatility of the process A over its continuous part, λ is the average number of jumps per year (intensity of jumps), γ the average of the relative price at jump times, and δ the standard deviation of the relative price at jump times.

Table 2: QML estimates of θ – Phar-Mor

Present date T	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\lambda}$	$\widehat{\gamma}$	$\widehat{\delta}$
Pure diffusion without frictions					
18 months before bankruptcy	-9.3%	28.6%	–	–	–
12 months before bankruptcy	-27.5%	29.0%	–	–	–
Jump diffusion without frictions					
18 months before bankruptcy	-3.0%	25.0%	6.15	2.0%	5.0%
12 months before bankruptcy	-24.0%	26.0%	6.20	1.5%	6.0%
Pure diffusion with frictions					
18 months before bankruptcy	-10.0%	30.0%	–	–	–
12 months before bankruptcy	-26.5%	31.0%	–	–	–
Jump diffusion with frictions					
18 months before bankruptcy	-5.0%	26.0%	7.05	1.4%	5.0%
12 months before bankruptcy	-26.0%	27.0%	7.88	1.0%	4.0%

Jumps tend to attenuate the (negative) drift and volatility parameters of the lognormal process. This is compensated by multiple jumps (per year) with small amplitudes and fluctuations. Table 3 exhibits the yield to maturity (YTM) of Phar-Mor's debt portfolio, its yield (credit) spread (YS), and its term structure of total default probabilities (TDP) under the physical probability measure for the horizons 6, 12, and 18 months at the present date T.

Table 3: Credit-risk indicators – Phar-Mor

Present date T	YTM	YS	TDP_6	TDP_{12}	TDP_{18}
Pure diffusion without frictions					
18 months before bankruptcy	7.31%	1.78%	4%	11%	20%
12 months before bankruptcy	8.24%	2.73%	26%	48%	70%
Jump diffusion without frictions					
18 months before bankruptcy	7.63%	2.10%	7%	15%	28%
12 months before bankruptcy	8.21%	2.70%	25%	52%	77%
Pure diffusion with frictions					
18 months before bankruptcy	9.12%	3.59%	3%	10%	20%
12 months before bankruptcy	12.08%	6.57%	22%	48%	74%
Jump diffusion with frictions					
18 months before bankruptcy	9.35%	3.82%	7%	17%	35%
12 months before bankruptcy	13.59%	8.08%	24%	54%	81%

Credit-risk indicators deteriorate when we approach bankruptcy, but those computed under the physical probability measure are more sensitive. This contradicts Delianedis and Geske (2003) who focus on changes in risk-neutral credit-risk measures. In line with the literature on the credit-spread puzzle, Phar-Mor's yield spreads are adjusted high when we count for frictions and jumps. They become more consistent with Moody's B3 rating of Phar-Mor from November 1995 to February 2001, as reported in Moody's Default & Recovery Data Base.

See Collin–Dufresne et al. (2001), Driessen (2005), Chen (2010), Bao et al. (2011), Gemmill and Keswani (2011), Huang and Huang (2012), and Du et al. (2019) for further details on the credit-spread puzzle. We share the idea that frictions and jumps partially explain the credit-spread puzzle. We also believe that credit spreads remain under-valued because market participants count for potential future debt issues, which are refuted by structural settings.

4 Conclusion

We propose a flexible and efficient QML approach to estimate complex structural settings. QML combines between DP and ML to simultaneously solve and estimate structural models. We conduct an empirical investigation of a public company that went bankrupt. We show that its credit-risk indicators deteriorate when we approach bankruptcy, while its credit spreads remain under-valued. Credit spreads are adjusted high when we experiment with frictions and jumps, which is consistent with the literature on the credit-spread puzzle.

A future research avenue consists of adjusting the design of structural settings to count for potential future debt issues, reworking QML, and revisiting the credit-risk puzzle.

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