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Payment schemes for sustaining cooperation in dynamic games

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Abstract : It is a challenge to sustain cooperation in a finite-horizon dynamic game. Since players generally have an incentive to deviate to their noncooperative strategies in the last stage, a backward induction argument leads them to defect from cooperation in all stages. In this paper, we propose two payment schemes having some desirable properties, namely, individual rationality and stability, which ensure that the players cooperate throughout the entire planning horizon. The setup and the results are general, that is, they do not rest on particular specifications of the payoff functionals or the state dynamics. We illustrate our results with a linear-quadratic dynamic game of pollution control.

Keywords: Dynamic games, sustainability of cooperation, payment schemes, individual rationality, efficiency

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1 Introduction

Solving a cooperative game with a transferable utility amounts to determining the best outcome that the grand coalition can achieve, by optimizing the (possibly weighted) sum of the players' payoffs, and allocating the optimal outcome to the players. The resulting strategy profile can be interpreted as the *operational clause* of the agreement (or contract), while the sharing represents the *financial clause*. In a one-shot game, both clauses are fulfilled simultaneously.

An additional concern comes up in a dynamic setting, namely, the durability (or sustainability) of an agreement over time. The players must ensure that the agreed-upon decisions will be indeed implemented as time goes by. As a cooperative strategy profile is typically not an equilibrium, i.e., the agreement is not self-enforced, a mechanism must be put in place to keep players from switching to their noncooperative strategies before the agreement reaches its maturity.

In this paper, we propose two allocation schemes that sustain a cooperative agreement until its deadline. Importantly, these schemes do not require any assumption about the structure of the dynamic game or the functional forms, and are based on the following two groups of properties:

1. Continuation of cooperation at any stage of the game;
2. Feasible payments over time.

The first group includes two properties, namely, *dynamic individual rationality* (DIR), i.e., the players are better off cooperating than playing noncooperatively in any subgame, and *stability against individual deviations* (SAID), i.e., cooperation is an equilibrium in any subgame. Dynamic individual rationality is often referred to as time consistency in the cooperative dynamic games literature, both in discrete and continuous time; see, e.g., the surveys/tutorials in Zaccour (2008, 2017), Petrosyan and Zaccour (2018), and Yeung and Petrosyan (2018). The stability against individual deviations is the main piece in making cooperation a subgame-perfect equilibrium in both repeated games and state-space dynamic games; see, e.g., Radner (1980), Benoit and Krishna (1985), and Tolwinski et al. (1986). It is well known that, in general, it is not possible to construct a cooperative equilibrium in finite-horizon games. Indeed, because the players have an incentive to defect from cooperation in the last stage, then, by a simple backward induction argument, they end up defecting in all stages. As an alternative, a subgame perfect ε-cooperative equilibrium can be considered, where ε is the maximum benefit a player can achieve by individually deviating from cooperation; see, e.g., Mailath et al. (2005), Parilina and Zaccour (2015a), and Flesch and Predtetchinski (2016). The payment schemes proposed in the paper are deviation-free, that is, $\varepsilon = 0$.

The second set of properties (feasibility) defines the payments to the players in any stage of the game, along the cooperative state trajectory. In most contributions to cooperative dynamic games, banking part of the realized current payoffs for future payments is not possible, that is, the total collective payoff must be fully allocated in each stage; see, e.g., Petrosjan and Danilov (1979), Yeung and Petrosyan (2012), Parilina and Zaccour (2015b, 2016), Kuzyutin et al. (2019), Dahmouni et al. (2019), and Gromova and Plekhanova (2019). We relax this constraint by allowing for savings for future use. In experimental games, players are willing, in early stages, to sacrifice some gains in order to keep cooperation running, but typically not towards the end of the game; see, e.g., Angelova et al. (2013) and Bruttel and Friehe (2014). In our case, the savings realized in the earliest stages are invested to boost the payments in last stages, which prevents deviation from cooperation.

The closest approach to ours is the so-called *imputation distribution procedure* (IDP). Defining an IDP involves three steps: First, a cooperative game in characteristic function (CF) form is defined for all subgames, including the whole game. Second, an agreed-upon imputation, e.g., the Shapley value or an imputation in the core, is computed for all subgames. Finally, the payments over stages are determined such that the players' payoffs-to-go in any subgame belong to the same solution concept, e.g., be the Shapley value in the cooperative subgame. The idea of an IDP was initially proposed for differential games in Petrosjan and Danilov (1979), and was later adapted to different classes of games, e.g., games on networks (Petrosyan and Sedakov (2016)), multicriteria (Kuzyutin et al. (2018)), and random planning horizon games (Gromova and Plekhanova (2018)).

There are two inherent difficulties in implementing an IDP, one conceptual, and the other computational. The conceptual difficulty arises from the multiplicity of CFs, with each giving different outcomes and having its pros and cons; see, e.g., Chander and Tulkens (1997), Germain et al. (2003), Petrosjan and Zaccour (2003), Reddy and Zaccour (2016), and Gromova and Petrosyan (2017). The implication is that choosing a CF becomes in itself a negotiation issue. The computational difficulty is in the determination, at each stage, of $2^m - 1$ characteristic function values, where m is the number of players. For instance, if the game involves 5 players and 10 stages, then 310 values must be computed. This is far from being a computationally friendly task, especially if these values correspond to equilibrium outcomes, the players are asymmetric and their payoff functions are highly nonlinear.

The difference between our payment schemes and an IDP is twofold. First, our payment schemes are stable against individual deviation, a property that is generally not fulfilled by an IDP, as shown in Petrosyan (2008) and Parilina and Tampieri (2018). Second, our payment schemes do not require the introduction of a CF, and consequently they escape the two above mentioned difficulties. Still, interestingly, we show that the proposed payments correspond to an imputation of a cooperative game in the γ-characteristic function (see Chander and Tulkens (1997)).

In the literature on gradual investment (see, e.g., Admati and Perry (1991) and Marx and Matthews (2000)), as in our paper, players behave cooperatively by choosing the investment level stage by stage to contribute to a joint or social project. The rewards are cashed when the project is finished. In this literature, the Nash equilibrium contribution policies and conditions to complete the project are found. Here, we start by finding the cooperative strategy profile that maximizes the total joint profit. Once this is done, the players' behavior along the state trajectory becomes given, but the rewards or payments are not fixed. The introduced payment schemes do not change players' strategies, but redefine the payoff functions to sustain cooperation and avoid possible players' deviations.

The remainder of the paper is organized as follows. In Section 2, we describe the dynamic game model, and in Section 3, we state some desirable properties of a payment scheme and provide several results on the relationships between these properties. The two payment schemes are defined and discussed in Section 4. We show that these payments are imputations of a cooperative game in Section 5. An illustrative example is discussed in Section 6, and we briefly conclude in Section 8.

2 Elements of the game

We consider a finite-horizon deterministic dynamic game played on $\mathbb{T} = \{0, 1, \dots, T\}$ and defined by the following elements[1]:

1. A set of players $M = \{1, 2, \dots, m\}$;
2. For each player $i \in M$, a vector of control (or decision) variables $u_i(t) \in U_i \subseteq \mathbb{R}^{m_i}$ at $t = 0, \dots, T-1$, where U_i is the set of admissible control values for Player i;
   Let $\mathbf{U} = \prod_{i \in M} U_i$;
3. A vector of state variables $x(t) \in X \subset \mathbb{R}^q$ at time $t \in \mathbb{T}$, where X is the set of admissible states and where the evolution over time of the state is given by

$$x(t+1) = f(t, x(t), u(t)), \quad x_0 \text{ given}, \tag{1}$$

   where $u(t) \in \mathbf{U}$, $t = 0, \dots, T-1$ and x_0 is the initial state at $t = 0$;

[1] See Basar and Olsder (1999) and Haurie et al. (2012) for the description of different classes of dynamic and differential games.

4. A payoff functional for Player $i \in M$,

$$J_i(\mathbf{u}; x_0) = \sum_{t=0}^{T-1} \rho^t \phi_i(t, x(t), u(t)) + \rho^T \Phi_i(x(T)), \tag{2}$$

where $\rho \in (0,1)$ is the discount factor; $u(t) = (u_1(t), \ldots, u_m(t))$, and $\mathbf{u}$ is given by

$$\mathbf{u} = \{u_i(t) \in U_i : t \in \mathbb{T}\backslash\{T\},\ i \in M\}; \tag{3}$$

$\phi_i(t, x(t), u(t))$ is the reward to Player i at $t = 0, \ldots, T-1$, and $\Phi_i(x(T))$ is the reward to Player i at terminal time T;

5. An information structure that defines the information that is available to Player $i \in M$ when she selects her control vector $u_i(t)$ at time $t \in \mathbb{T}\backslash\{T\}$;
6. A strategy η_i for Player $i \in M$, which is an m_i-dimensional vector-decision rule that defines the control $u_i(t) \in U_i$ as a function of the information available at time $t = 0, \ldots, T-1$.

If the players agree to cooperate, then they will maximize their joint payoff

$$J(\mathbf{u}; x_0) = \sum_{i \in M} J_i(\mathbf{u}; x_0). \tag{4}$$

Let $u^*(t) = (u_1^*(t), \ldots, u_m^*(t)), t = 0, \ldots, T-1$ be the control paths that solve the optimal control problem (4), subject to the state Equations (1). Denote by $J^*(x_0)$ the outcome of the joint optimization problem. If this solution is implemented throughout the game, then Player i gets the following before-side-payment outcome (BSPO):

$$J_i^*(x_0) = \sum_{t=0}^{T-1} \rho^t \phi_i(t, x^*(t), u^*(t)) + \rho^T \Phi_i(x^*(T)), \tag{5}$$

where $x^*(t)$ is the solution to the state equation

$$x(t+1) = f(t, x(t), u^*(t)), \quad x_0 \text{ given.} \tag{6}$$

If one interprets the cooperative solution as a contract between the players, then the control paths $u^*(t), t = 0, \ldots, T-1$ represent the operational clause of this contract. That is, each player agrees to implement, at each period t, the action that realizes the collectively optimal payoff

$$J^*(x_0) = \sum_{i \in M} J_i^*(x_0).$$

Further, the BSPO of Player i may not be larger than the outcome she can secure by acting alone. Therefore, the contract must also include a financial clause specifying the payments that the players will receive at each period. For such a clause to be agreeable to all players, it must satisfy some requirements, e.g., individual rationality and fairness. In Section 4, we introduce a payment scheme that has desirable properties.

The alternative (and benchmark) to cooperation is a noncooperative mode of play. In such event, the players seek a feedback Nash equilibrium, in which Player i gets the following outcome:

$$J_i^{nc}(x_0) = \sum_{t=0}^{T-1} \rho^t \phi_i(t, x^{nc}(t), u^{nc}(t)) + \rho^T \Phi_i(x^{nc}(T)), \tag{7}$$

where $x^{nc}(t)$ and $u^{nc}(t), t = 0, \ldots, T-1$ are the state and control trajectories, respectively.

We make the following assumptions:

A1: If cooperation breaks down at any intermediate period $\tau > 0$, i.e., the strategy profile in τ is different from the cooperative one $u^*(\tau)$, then the players switch to their Nash equilibrium strategies in the remaining subgame (in periods $\tau+1$, $\tau+2$, …, $T-1$) with the initial state being $x^*(\tau)$.

A2: There exists a unique feedback Nash equilibrium in each subgame, or a device for selecting one equilibrium if there are many.

The first assumption is quite natural and typical in the literature on cooperative dynamic games; see, e.g., Petrosyan and Zaccour (2018). It simply states that the players start by cooperating. The second assumption is meant to avoid equilibrium selection, an issue that is well beyond the objective of this paper.

Remark 1. The uniqueness of the joint-optimization solution requires, as usual, strict concavity of the objective function and the control set must be compact and convex. When the multistage game has a normal form representation, the conditions for uniqueness of Nash equilibrium are the same as for games with continuous payoffs with constraints as established in Rosen (1965). For the class of linear-quadratic dynamic games, which is widely used in applications, the conditions of the existence and uniqueness of the Nash equilibrium in open-loop and feedback information structures can be found in Basar and Olsder (1999), Jank and Abou-Kandil (2003).

3 Properties of a payment scheme

Let $u_{-i}(t) = (u_1(t), \ldots, u_{i-1}(t), u_{i+1}(t), \ldots, u_m(t))$. Denote by $p_i(t)$ the payment that Player $i \in M$ receives in period $t \in \mathbb{T}$ under the cooperative mode of play. We shall refer to the vector $(p_i(t) : i \in N, t \in \mathbb{T})$ as a payment scheme $\mathcal{P}$. In practice, any payment scheme is based on some desirable properties. Here, we state the following six properties, and then define schemes that satisfy different combinations of them:

P1: Feasibility. $\mathcal{P}$ is *feasible* if

$$\sum_{i\in M} p_i(t) \leq \sum_{i\in M} \phi_i(t, x^*(t), u^*(t)) + \sum_{\tau=0}^{t-1} \frac{1}{\rho^{t-\tau}} \sum_{i\in M} \Big(\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)\Big), \tag{8}$$

for all $t \in \mathbb{T} \setminus \{T\}$, and

$$\sum_{i\in M} p_i(T) \leq \sum_{i\in M} \Phi_i(x^*(T)) + \sum_{\tau=0}^{T-1} \frac{1}{\rho^{T-\tau}} \sum_{i\in M} \Big(\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)\Big) \tag{9}$$

for $t = T$.

P2: Dynamic individual rationality (DIR). $\mathcal{P}$ is *dynamically individually rational* if

$$\sum_{\tau=t}^{T} \rho^{\tau-t} p_i(\tau) \geq J_i^{nc}(t, x^*(t)), \tag{10}$$

for all $t \in \mathbb{T}$ and all $i \in M$, where $J_i^{nc}(t, x^*(t))$ is the Nash equilibrium outcome of Player i in the subgame starting at period t with state $x^*(t)$.

P3: Stability against individual deviation (SAID). $\mathcal{P}$ is *stable against individual deviation* if

$$\begin{aligned}\sum_{\tau=t}^{T} \rho^{\tau-t} p_i(\tau) &\geq \max_{u_i(t)\in U_i} \left\{\phi_i\left(t, x^*(t), (u_{-i}^*(t), u_i(t))\right) + \rho J_i^{nc}(t+1, \hat{x}(t+1))\right\} \\ &:= BR_i(t, x^*(t)),\end{aligned} \tag{11}$$

for all $t \in \mathbb{T}$ and all $i \in M$, where $BR_i(T, x^*(T)) := \Phi_i(x^*(T))$ and

$$\hat{x}(t+1) = f(t, x^*(t), \left(u^*_{-i}(t), \hat{u}_i(t)\right)),$$

where $\hat{u}_i(t)$ is Player i's optimal control that solves the maximization problem in (11).

P4: Efficiency. $\mathcal{P}$ is *efficient* if

$$\sum_{i\in M}\sum_{t=0}^{T}\rho^t p_i(t) = \sum_{i\in M} J_i^*(x_0). \tag{12}$$

P5: Stage budget balance (SBB). $\mathcal{P}$ is *stage budget balanced* if

$$\sum_{i\in M} p_i(t) = \sum_{i\in M}\phi_i\left(t, x^*(t), u^*(t)\right), \quad t\in\mathbb{T}\setminus\{T\}, \tag{13}$$

$$\sum_{i\in M} p_i(T) = \sum_{i\in M}\Phi_i\left(x^*(T)\right). \tag{14}$$

P6: Minimal required savings (MRS). $\mathcal{P}$ satisfies the property of *minimal required savings* if

$$\begin{aligned}\sum_{\tau=0}^{t}\frac{1}{\rho^{t-\tau}}\sum_{i\in M}\Big(\phi_i\left(\tau, x^*(\tau), u^*(\tau)\right) - p_i(\tau)\Big) = &-\sum_{\tau=t+1}^{T-1}\rho^{\tau-t}\sum_{i\in M}\Big(\phi_i\left(\tau, x^*(\tau), u^*(\tau)\right) - p_i(\tau)\Big)\\ &-\rho^{T-t}\sum_{i\in M}\Big(\Phi_i\left(x^*(T)\right) - p_i(T)\Big)\end{aligned} \tag{15}$$

for all $t \in \mathbb{T}\setminus\{T\}$.

These properties deserve few comments. In (8), the first term on the right-hand side (RHS) is the total collective reward at time t. The difference $\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)$ represents the *shortage* that Player i consents to at time τ. (Note that this shortage can take any sign.) So, the second term on the RHS of (8) is the capitalized value of the sum of shortages. Therefore, the condition in (8) states that the total payments at t should not exceed the total cooperative reward at t plus this capitalized value. In particular, at $t = 0$, this condition, which takes the form

$$\sum_{i\in M} p_i(0) \le \sum_{i\in M}\phi_i\left(0, x_0, u^*(0)\right),$$

requires that the total payments do not exceed the total optimal collective payoff at that period. The difference $\sum_{i\in M}\phi_i(0, x_0, u^*(0)) - \sum_{i\in M} p_i(0)$ represents the saving that is invested for later use.

For the payment scheme to be individually rational at any time t, the payment-to-go to any player i, given by the left-hand side (LHS) of (10), must be at least equal to her noncooperative payoff-to-go in the subgame starting from the cooperative state $x^*(t)$ (RHS of (10)).

The stability against individual deviation of the payment scheme $\mathcal{P}$ means that the cooperative payoff-to-go of Player i, in the subgame starting at (any) t with initial state value $x^*(t)$ (LHS of (11)), is at least equal to what she can obtain by deviating unilaterally. The control $\hat{u}_i(t)$ is the best reply (hence, the notation $BR_i(t, x^*(t))$) of Player i to $u^*_{-i}(t)$. By Assumption A1, after the deviation is identified, the game is played noncooperatively from $t+1$ onward, with the initial state given by $\hat{x}(t+1) = f\left(t, x^*(t), (u^*_{-i}(t), \hat{u}_i(t))\right)$ (RHS in inequality (11)).

Efficiency guarantees that the total of the discounted payments to the players is equal to the total joint-maximization payoff. That is, subsidies and wastages are ruled out under $\mathcal{P}$. The SBB property stipulates that the stage's optimal joint payoff be fully allocated to the players. Finally, the MRS property means that the total capitalized savings at any time t is equal to what is required to implement the payment scheme in the rest of the game.

We note that DIR and SAID imply that

$$\sum_{\tau=t}^{T} \rho^{\tau-t} p_i(\tau) \geq \max\left\{J_i^{nc}(t, x^*(t)), BR_i(t, x^*(t))\right\}. \tag{16}$$

Further, the inequalities (8), (9), (10), and (11) would have the opposite sign if the players minimized their objective functionals.

The following propositions establish some relationships between feasibility, SBB, efficiency and MRS.

Proposition 1. *If a payment scheme $\mathcal{P} = (p_i(t) : i \in M, t \in \mathbb{T})$ is feasible, then*

$$\sum_{i \in M} \sum_{t=0}^{T} \rho^t p_i(t) \leq \sum_{i \in M} J_i^*(x_0). \tag{17}$$

Proof. Assume $\mathcal{P}$ is feasible and consider the difference

$$\begin{aligned}
\sum_{i \in M} \sum_{t=0}^{T} \rho^t p_i(t) - \sum_{i \in M} J_i^*(x_0) &= \sum_{t=0}^{T-1} \rho^t \sum_{i \in M} (p_i(t) - \phi_i(t, x^*(t), u^*(t))) + \rho^T \sum_{i \in M} (p_i(T) - \Phi_i(x^*(T))) \\
&\leq \sum_{t=0}^{T-1} \rho^t \sum_{i \in M} (p_i(t) - \phi_i(t, x^*(t), u^*(t))) + \rho^T \sum_{t=0}^{T-1} \frac{1}{\rho^{T-t}} \sum_{i \in M} \Big(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\Big) \\
&= 0,
\end{aligned}$$

which proves (17). □

Proposition 2. *If a payment scheme is stage-budget balanced, then it satisfies the Efficiency, Feasibility, and MRS properties.*

Proof. Suppose $\mathcal{P}$ is SBB. Computing $\sum\limits_{i \in M} \sum\limits_{t=0}^{T} \rho^t p_i(t)$, we get

$$\begin{aligned}
\sum_{i \in M} \sum_{t=0}^{T} \rho^t p_i(t) &= \sum_{t=0}^{T} \rho^t \sum_{i \in M} p_i(t) = \sum_{t=0}^{T-1} \rho^t \sum_{i \in M} \phi_i(t, x^*(t), u^*(t)) + \rho^T \sum_{i \in M} \Phi_i(x^*(T)) \\
&= \sum_{i \in M} J_i^*(x_0),
\end{aligned}$$

which shows Efficiency.

If $\mathcal{P}$ is SBB, then the feasibility condition in (8) becomes

$$\sum_{i \in M} p_i(t) \leq \sum_{i \in M} \phi_i(t, x^*(t), u^*(t)),$$

and in (9) becomes

$$\sum_{i \in M} p_i(T) \leq \sum_{i \in M} \Phi_i(x^*(T)),$$

which hold with equality for an SBB scheme.

Finally, if $\mathcal{P}$ is SBB, then for any $t \in \mathbb{T} \setminus \{T\}$, we have

$$\sum_{i \in M} \Big(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\Big) = 0,$$

which implies that the equality in (15) is trivially satisfied. □

Proposition 3. *If a payment scheme satisfies the minimal required savings property, then it is efficient.*

Proof. If $\mathcal{P}$ satisfies MRS, then at $t = 0$, (15) becomes

$$\sum_{i\in M}\Big(\phi_i\left(0, x_0, u^*\left(0\right)\right) - p_i(0)\Big) = -\sum_{\tau=1}^{T-1}\rho^{\tau}\sum_{i\in M}\Big(\phi_i\left(\tau, x^*\left(\tau\right), u^*\left(\tau\right)\right) - p_i(\tau)\Big) \\ - \rho^T\sum_{i\in M}\Big(\Phi_i\left(x^*\left(T\right)\right) - p_i(T)\Big),$$

that can be rewritten as

$$\sum_{i\in M}J_i^*(x_0) = \sum_{i\in M}\sum_{\tau=0}^{T}\rho^{\tau}\phi_i\left(\tau, x^*\left(\tau\right), u^*\left(\tau\right)\right) + \rho^T\sum_{i\in M}\Phi_i\left(x^*\left(T\right)\right) = \sum_{i\in M}\sum_{t=0}^{T}\rho^t p_i(t),$$

which proves that $\mathcal{P}$ is efficient. □

We add three observations: First, obviously, Feasibility does not imply MRS. Second, an efficient payment scheme does not necessarily satisfy MRS. Finally, an efficient payment scheme may not be feasible. To illustrate, consider the following simple scheme:

$$p_i'(t) = \begin{cases} J_i^*(x_0), & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is clearly efficient but not feasible if, for at least one time $t = 1, \ldots, T-1$, we have

$$\sum_{i\in M}\phi_i\left(t, x^*\left(t\right), u^*\left(t\right)\right) > 0.$$

4 Payment schemes

To start, we define a (preliminary) payment scheme that satisfies the Feasibility, DIR and SAID properties.

Definition 1. A feasible payment scheme that satisfies

$$\sum_{\tau=t}^{T}\rho^{\tau-t}p_i(\tau) = \max\{J_i^{nc}(t, x^*\left(t\right)); BR_i(t, x^*\left(t\right))\}, \text{ for any } t \in \mathbb{T},$$

is called a minimal payment scheme and is denoted $\mathcal{P}_{\min}$.

Put differently, if it exists, a scheme $\mathcal{P}_{\min}$ satisfies the feasibility property, and DIR holds with equality, which implies the SAID property if $J_i^{nc}(t, x^*\left(t\right)) \geq BR_i(t, x^*\left(t\right))$, or SAID holds with equality, implying the DIR property if $J_i^{nc}(t, x^*\left(t\right)) < BR_i(t, x^*\left(t\right))$. A $\mathcal{P}_{\min}$ scheme guarantees that the players have no incentive to deviate along the cooperative state trajectory.

To construct $\mathcal{P}_{\min}$, we proceed backward from the terminal time T, simultaneously satisfying DIR and SAID, and next checking if Feasibility is verified. The steps are as follows:

1. Let $t = T$. Set
$$p_i(T) := \Phi_i(x_T^*), \quad \text{for all } i \in M. \tag{18}$$
Properties DIR and SAID are trivially satisfied for $t = T$.

2. Let $t = T-1$. For all $i \in M$, set

$$\begin{aligned} p_i(T-1) &:= \max\{J_i^{nc}(T-1, x^*(T-1)) - \rho p_i(T); BR_i(T-1, x^*(T-1)) - \rho p_i(T)\} \\ &= \max\{J_i^{nc}(T-1, x^*(T-1)); BR_i(T-1, x^*(T-1))\} - \rho p_i(T). \end{aligned} \tag{19}$$

   Properties DIR and SAID are satisfied by construction for $t = T-1$.

3. Consider any period $t = T-2, \ldots, 0$. Define the payments $p_i(t)$, $i \in M$, as

$$p_i(t) := \max\{J_i^{nc}(t, x^*(t)); BR_i(t, x^*(t))\} - \sum_{\tau=t+1}^{T} \rho^{\tau-t} p_i(\tau). \tag{20}$$

   Properties DIR and SAID are satisfied by construction for $t \in \{T-2, \ldots, 0\}$.

4. Check the Feasibility condition, that is, the inequalities in (8) and (9). If it is satisfied for all $t \in \mathbb{T}$, then a $\mathcal{P}_{\min}$ exists and is defined by (18)–(20); otherwise, no $\mathcal{P}_{\min}$ exists.

If a $\mathcal{P}_{\min}$ exists, then we can construct a first payment scheme that satisfies the Feasibility, DIR, SAID, and Efficiency properties, and a second one that additionally satisfies MRS.

Remark 2. In the next two propositions, we introduce two different payment schemes, assuming that a minimal payment scheme exists. A natural question is what can be done if this isn't the case? Two approaches can be followed: (i) drop either the DIR or SAID property, which implies a decrease in the payments to players at all stages, but the last; or, (ii) determine the range of the discount factor for which a minimal payment scheme exists.

4.1 A first payment scheme

The construction of the first payment scheme $\mathcal{P}_1$ is based on the following proposition and its constructive proof:

Proposition 4. *If a $\mathcal{P}_{\min} = (p_i(t) : i \in M, t \in \mathbb{T})$ exists, then the following payment scheme exists and satisfies the Feasibility, DIR, SAID, and Efficiency properties:*

$$\mathcal{P}_1 \begin{cases} \text{For } t = 0, \ldots, T-1, \\ p_i'(t) := p_i(t). \\ \\ \text{For } t = T, \\ p_i'(T) := p_i(T) + \sigma_i(T), \\ \text{with } \sigma_i(T) \geq 0 \text{ for any } i \in M, \\ \text{and} \sum_{i\in M} \sigma_i(T) = \sum_{\tau=0}^{T-1} \frac{1}{\rho^{T-\tau}} \sum_{i\in M} \Big(\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)\Big). \end{cases} \tag{21}$$

Proof. Suppose $\mathcal{P}_{\min}$ exists. Then, it is uniquely defined by (18)–(20). Now, we construct an efficient payment scheme that satisfies the Feasibility, DIR, and SAID properties.

$\mathcal{P}_{\min} = (p_i(t) : i \in M, t \in \mathbb{T})$ satisfies Feasibility, so for any $t \in \mathbb{T} \setminus \{T\}$ we have

$$\sum_{i\in M} p_i(t) \leq \sum_{i\in M} \phi_i(t, x^*(t), u^*(t)) + \sum_{\tau=0}^{t-1} \frac{1}{\rho^{t-\tau}} \sum_{i\in M} \Big(\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)\Big),$$

and for $t = T$ we have

$$\sum_{i\in M} p_i(T) \leq \sum_{i\in M} \Phi_i(x^*(T)) + \sum_{\tau=0}^{T-1} \frac{1}{\rho^{T-\tau}} \sum_{i\in M} \Big(\phi_i(\tau, x^*(\tau), u^*(\tau)) - p_i(\tau)\Big),$$

Now, define the new payments

$$p_i'(t) = \begin{cases} p_i(t), & \text{if } t \in \mathbb{T} \setminus \{T\}, \\ p_i(t) + \sigma_i(t), & \text{if } t = T, \end{cases} \tag{22}$$

where $\sigma_i(T)$ is such that $\sigma_i(T) \geq 0$ for any $i \in M$ and

$$\sum_{i\in M} \sigma_i(T) = \sum_{\tau=0}^{T-1} \frac{1}{\rho^{T-\tau}} \sum_{i\in M} \Big(\phi_i\left(\tau, x^*\left(\tau\right), u^*\left(\tau\right)\right) - p_i(\tau)\Big).$$

The above equation means that the players redistribute the savings by time T at this period according to some allocation rule. Therefore, the payments $p_i'(t)$, for $i \in M$ coincide with those defined by $\mathcal{P}_{\min}$, but at the terminal time. As $p_i'(t) = p_i(t)$, $t \in \mathbb{T} \setminus \{T\}$, then the DIR and SAID properties are obviously satisfied. As Feasibility is also satisfied by construction, it remains to show Efficiency, i.e., to have

$$\sum_{i\in M}\sum_{t=0}^{T} \rho^t p_i'(t) = \sum_{i\in N} J_i^*(x_0). \tag{23}$$

Substituting for $p_i'(t)$ from (22) into the LHS of (23), we easily obtain that

$$\sum_{i\in M}\sum_{t=0}^{T} \rho^t p_i'(t) = \sum_{t=0}^{T-1} \rho^t \sum_{i\in M} \phi_i\left(t, x^*\left(t\right), u^*\left(t\right)\right) + \rho^T \sum_{i\in M} \Phi_i\left(x^*\left(T\right)\right) = \sum_{i\in M} J_i^*(x_0).$$

□

We make the following observations: First, compared to the minimal payment scheme, $\mathcal{P}_1$ is efficient ($\mathcal{P}_{\min}$ is not). Second, the payment scheme defined above is not unique, because the vector $\sigma\left(T\right) = \left(\sigma_1(T), \ldots, \sigma_m(T)\right)$ is not uniquely defined. Third, the existence of $\mathcal{P}_{\min}$, which implies the existence of $\mathcal{P}_1$, depends on the discount factor ρ. The lower the value of ρ, the less savings are attractive. This result is straightforward to interpret through the lens of the folk theorem, i.e., for cooperation (or collusion) to be sustained, the discount factor must be high enough. Finally, the early-stage savings used to make the final payment ensure the sustainability of the agreement till the terminal date.

4.2 A second payment scheme

The first payment scheme requires that the savings only be distributed at the terminal date. Now, we relax this obligation by allowing the allocation of savings over time along the cooperative state trajectory. The second payment scheme $\mathcal{P}_2$, defined in the next proposition, satisfies the additional MRS property.

For $t = 1, \ldots, T-1$, denote by $B(t)$ the amount of savings needed to make payments in period t and future time periods. This amount is defined by

$$B(t) := \left[\rho B(t+1) + \sum_{i\in M} \left(p_i(t) - \phi_i\left(t, x^*\left(t\right), u^*(t)\right)\right)\right]^+, \tag{24}$$

that is, $B\left(t\right)$ is equal to the RHS expression if it is positive, and zero otherwise. In (24), $p_i(t)$ is a payment of a minimal payment schemed defined by (18)–(20). We let $B\left(T\right) := 0$. For all $i \in M$ and $t = 0, \ldots, T-1$, denote by $\xi_i\left(t\right)$ the nonnegative number satisfying

$$\sum_{i\in M} \xi_i(t) = \sum_{i\in M} \left(\phi_i(t, x^*\left(t\right), u^*(t)) - p_i(t)\right) - \rho B(t+1).$$

Recall that $p_i(t)$ is the payment in $\mathcal{P}_{\min}$ uniquely defined by (18)–(20).

Proposition 5. *If a $\mathcal{P}_{\min} = (p_i(t) : i \in M, t \in \mathbb{T})$ exists, then the following payment scheme exists and satisfies Feasibility, DIR, SAID, Efficiency, and MRS:*

$$\mathcal{P}_2 \quad \begin{cases} \textit{For } t = T: \\ p_i''(t) := p_i(T); \\ \\ \textit{For } t = T-1, \ldots, 1: \\ p_i''(t) = \begin{cases} p_i(t), & \textit{if } \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1) < 0, \\ p_i(t) + \xi_i(t), & \textit{if } \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1) \geq 0, \end{cases} \\ \textit{where } \xi_i(t) \geq 0 \; s.t. \; \sum\limits_{i \in M} \xi_i(t) = \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1); \\ \textit{For } t = 0: \\ p_i''(0) = p_i(0) + \xi_i(0), \\ \textit{where } \xi_i(0) \geq 0 \; s.t. \; \sum\limits_{i \in M} \xi_i(0) = \sum\limits_{i \in M} \left(\phi_i(0, x_0, u^*(0)) - p_i(0)\right) - \rho B(1), \end{cases}$$

where $B(t)$ is recurrently defined by (24) with terminal condition $B(T) = 0$.

Proof. Suppose that $\mathcal{P}_{\min}$ exists and is defined by (18)–(20). We construct a payment scheme $\mathcal{P}_2$ backwards from $t = T$ to $t = 0$ as follows:

$t = T:$ At the terminal time, set $p_i''(T) = p_i(T)$, i.e., the payment to any player is the same as in $\mathcal{P}_{\min}$. This implies

$$B(T) := 0.$$

$t = T-1, \ldots, 1$**:** Set

$$p_i''(t) = \begin{cases} p_i(t), & \text{if } \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1) < 0, \\ p_i(t) + \xi_i(t), & \text{if } \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1) \geq 0, \end{cases}$$

where $\xi_i(t) \geq 0$ is such that $\sum\limits_{i \in M} \xi_i(t) = \sum\limits_{i \in M} \left(\phi_i(t, x^*(t), u^*(t)) - p_i(t)\right) - \rho B(t+1)$.

$t = 0$**:** By the feasibility property, which is satisfied for the minimal payment scheme, we have

$$\sum_{i \in M} \left(\phi_i(0, x_0, u^*(0)) - p_i(0)\right) - \rho B(1) \geq 0.$$

The payment to Player i at time 0 is given by

$$p_i''(0) = p_i(0) + \xi_i(0),$$

with $\xi_i(0) \geq 0$ subject to

$$\sum_{i \in M} \xi_i(0) = \sum_{i \in M} \left(\phi_0(0, x_0, u^*(0)) - p_i(0)\right) - \rho B(1).$$

By construction, the MRS property is satisfied, which implies Efficiency by Proposition 3. We need to prove Feasibility, DIR, and SAID. At any time t, the proposed payment $p''(t)$ is equal to or greater than $p(t)$ defined in the $\mathcal{P}_{\min}$ for which the DIR and SAID properties are satisfied. Now we prove Feasibility. Consider the Feasibility property for $t = 0$:

$$\sum_{i \in M} p_i''(0) \leq \sum_{i \in M} \phi_i\left(0, x_0, u^*(0)\right),$$

which is obviously satisfied because $p_i''(0) = p_i(0) + \xi_i(0)$, where

$$\sum_{i \in M} \left(p_i(0) + \xi_i(0)\right) \leq \sum_{i \in M} \phi_i(0, x_0, u^*(0))$$

by construction. Then consider $t = 1$: we have

$$p_i''(1) = \begin{cases} p_i(1), & \text{if } \sum\limits_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2) < 0, \\ p_i(1) + \xi_i(1), & \text{if } \sum\limits_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2) \geq 0, \end{cases}$$

where $\xi_i(1) \geq 0$ s.t. $\sum\limits_{i\in M} \xi_i(1) = \sum\limits_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2)$.

When $\sum\limits_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2) < 0$, the payment $p_i''(1)$ equals the payment in the minimal scheme, and Feasibility is satisfied as the minimal scheme is feasible.

If $\sum\limits_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2) \geq 0$, then we prove that Feasibility property is also satisfied for $t = 1$. We need to prove this:

$$\sum_{i\in M} p_i''(1) \leq \sum_{i\in M} \phi_i(1, x^*(1), u^*(1)) + \frac{1}{\rho} \sum_{i\in M} \Big(\phi_i(0, x^*(0), u^*(0)) - p_i''(0)\Big).$$

Substituting the expressions of $p_i''(1)$ and $p_i''(0)$, we can rewrite the last inequality as follows:

$$\sum_{i\in M} (p_i(1) + \xi_i(1)) \leq \sum_{i\in M} \phi_i(1, x^*(1), u^*(1)) + \frac{1}{\rho} \sum_{i\in M} \Big(\phi_i(0, x^*(0), u^*(0)) - (p_i(0) + \xi_i(0))\Big),$$

or equivalently

$$\sum_{i\in M} \xi_i(1) - \sum_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) \leq \frac{1}{\rho} \sum_{i\in M} \Big(\phi_i(0, x^*(0), u^*(0)) - (p_i(0) + \xi_i(0))\Big),$$

and substituting expression for $\sum_{i\in M} \xi_i(1)$, that is,

$$\sum_{i\in M} \xi_i(1) = \sum_{i\in M} (\phi_i(1, x^*(1), u^*(1)) - p_i(1)) - \rho B(2),$$

and we obtain

$$-\rho B(2) \leq \frac{1}{\rho} \sum_{i\in M} \Big(\phi_i(0, x^*(0), u^*(0)) - (p_i(0) + \xi_i(0))\Big),$$

which is true, as the expression in the LHS is nonpositive and the one in the RHS is nonnegative. We can easily prove Feasibility for any $t = 2, \dots, T$ in the same way.

Therefore, for the scheme defined by $p''(t)$, all these properties (Feasibility, DIR, SAID, Efficiency, and MRS) are satisfied. □

To interpret $\mathcal{P}_2$, let

$$\Delta_t \triangleq \left[\sum_{i\in M} (\phi_i(t, x^*(t), u^*(t)) - p_i(t))\right] - \rho B(t+1),$$

where $p_i(t)$ is defined by (18)–(20). The first square-bracketed term represents the difference between the realized total optimal reward and the total payments at t. The second term ($\rho B(t+1)$) is the discounted value of the amount available for borrowing at the next period. If Δ_t is negative, then set the payment to Player i at time t at its minimum value, in a $\mathcal{P}_{\min}$ sense, that is, $p_i''(t) = p_i(t)$. If $\Delta_t \geq 0$, then add a nonnegative allocation $\xi_i(t)$, with $\sum\limits_{i\in M} \xi_i(t) = \Delta_t$.

As for $\mathcal{P}_1$, we note that the payment scheme $\mathcal{P}_2$ is not unique. Again, the reason is that the vector $\xi(t) = (\xi_1(t), \dots, \xi_m(t))$ is not uniquely defined for any period t. Indeed, the players can choose any procedure to share the savings.

Remark 3. The main difference between the two payment schemes is that $\mathcal{P}_2$ satisfies the Minimal required savings property ($P6$), but $\mathcal{P}_1$ does not. Property $P6$ requires to determine the payments along the cooperative trajectory, which implies that $\mathcal{P}_2$ is more demanding computationally speaking than $\mathcal{P}_1$. On the other hand, the players gets more in the earlier periods under $\mathcal{P}_2$ than under $\mathcal{P}_1$. The timing of payments may be an important attribute in some contexts, e.g., when the players are involved in different projects and revenues from the one under consideration are needed for the other projects.

We illustrate the construction of the two payment schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ with a finitely repeated Prisoner's Dilemma game. Clearly, it is a simplified version of the dynamic game defined in Section 2, because no state dynamics are involved in a repeated game. Still, the example allows us to show the computations involved in $\mathcal{P}_1$ and $\mathcal{P}_2$, and highlight the impact of some parameter values on the existence of the minimal payment scheme.

Example 1. Consider the two-player Prisoner's Dilemma (PD) with payoff matrix

		Player 2	
		Deny	Confess
Player 1	Deny	(a,a)	(c,b)
	Confess	(b,c)	(d,d)

with $b > a > d > c$ and $2a > c + b$.

Let the PD be repeated in periods $\mathbb{T} = \{1,\dots,T\}$, and let ρ be the common discount factor. Obviously, the cooperative strategy profile consists in both players choosing strategy "Deny" in any period $t \in \mathbb{T}$. We denote this profile by $\mathbf{u}^* = (\mathbf{u}_1^*, \mathbf{u}_2^*)$, where $\mathbf{u}_i^* = (u_i^*(t) = \text{"Deny"}, t \in \mathbb{T})$, $i = 1,2$. Player i's cooperative payoff is given by

$$J_i^* = a\left(1+\rho+\rho^2+\ldots+\rho^{T-1}\right) = a\frac{1-\rho^T}{1-\rho}, \quad i=1,2.$$

The Nash equilibrium (which is also subgame perfect) strategy profile is $\mathbf{u}^{nc} = (\mathbf{u}_1^{nc}, \mathbf{u}_2^{nc})$, where $\mathbf{u}_i^{nc} = (u_i^{nc}(t) = \text{"Confess"}, t \in \mathbb{T})$, $i = 1,2$. Player i's payoff in the Nash equilibrium is

$$J_i^{nc} = d\left(1+\rho+\rho^2+\ldots+\rho^{T-1}\right) = d\frac{1-\rho^T}{1-\rho}, \quad i=1,2.$$

The Nash equilibrium payoff in a subgame starting in any t is

$$J_i^{nc}(t) = d\left(1+\rho+\rho^2+\ldots+\rho^{T-t}\right) = d\frac{1-\rho^{T-t+1}}{1-\rho}, \quad i=1,2.$$

Now, we compute $BR_i(t)$, which is the maximal payoff that player i can achieve by one-stage deviation in period t. Taking into account assumption A1, we obtain

$$BR_i(t) = b + d\left(\rho+\rho^2+\ldots+\rho^{T-t}\right) = b + d\frac{\rho-\rho^{T-t+1}}{1-\rho} = b + d\frac{\rho(1-\rho^{T-t})}{1-\rho}, \quad i=1,2.$$

Clearly, $BR_i(t) > J_i^{nc}(t)$ for any time $t = 1,\dots,T$ and $i = 1,2$.

First, we find the conditions under which the minimal payment scheme $\mathcal{P}_{\min}$ exists. In order to do this, we calculate the payments by (18), (19) and (20). (We correct for the fact that here the players also choose strategies at the terminal stage, whereas they do not in our game setting given in Section 2.) The payments defining $\mathcal{P}_{\min}$ are

$$\begin{aligned}
p_i(T) &= \max\{BR_i(T), J_i^{nc}(T)\} = BR_i(T) = b,\\
p_i(T-1) &= \max\{BR_i(T-1), J_i^{nc}(T-1)\} - \rho p_i(T) = b + \rho d - \rho b = (1-\rho)b + \rho d,\\
p_i(T-2) &= \max\{BR_i(T-2), J_i^{nc}(T-2)\} - \rho p_i(T-1) - \rho^2 p_i(T) = (1-\rho)b + \rho d,\\
p_i(t) &= (1-\rho)b + \rho d,\ t = T-3,\dots,1.
\end{aligned}$$

We verify if this payment scheme is feasible and check inequality (8) for any period $t = 1,\dots,T$:

For $t = 1$**:** $2((1-\rho)b + \rho d) \leq 2a$, which is equivalent to

$$\rho \geq \frac{b-a}{b-d}. \tag{25}$$

For $t = 2, \ldots, T-1$**:** the feasibility condition is

$$2((1-\rho)b + \rho d - a) \leq \frac{1}{\rho}2(a-(1-\rho)b-\rho d) + \ldots + \frac{1}{\rho^{t-1}}2(a-(1-\rho)b-\rho d),$$

which is equivalent to

$$((1-\rho)b + \rho d - a)\left(1 + \frac{1}{\rho} + \ldots + \frac{1}{\rho^{t-1}}\right) \leq 0,$$

and is satisfied for any $t = 2, \ldots, T-1$ when (25) is true.

For $t = T$**:** the feasibility condition is

$$2(b-a) \leq \frac{1}{\rho}2(a-(1-\rho)b-\rho d) + \ldots + \frac{1}{\rho^{T-1}}2(a-(1-\rho)b-\rho d),$$

which is equivalent to

$$\frac{\rho-\rho^T}{1-\rho^T} \geq \frac{b-a}{b-d}. \tag{26}$$

As $\frac{\rho-\rho^T}{1-\rho^T} \leq \rho$ for any $\rho \in (0,1)$, then condition (26) is stronger than (25). So, for feasibility of the payment scheme, condition (26) should be satisfied.

Moreover, $\frac{\rho-\rho^T}{1-\rho^T} \to \rho$ when $T \to \infty$. We can easily notice that the larger the duration of the game, the smaller discount factor that is required to satisfy the feasibility condition (26). For example, if $b = 15$, $a = 10$, $d = 4$, we have the following conditions for existence of the minimal payment scheme depending on the duration of the game:

$T=2$	$T=3$	$T=4$	$T=5$	$T=9$	$T\to\infty$
$\rho \geq 0.83$	$\rho \geq 0.54$	$\rho \geq 0.48$	$\rho \geq 0.47$	$\rho \geq 0.455$	$\rho \geq 0.4545$

If $b = 15$, $a = 12$, $d = 11$, we have the following conditions for existence of the minimal payment scheme depending on the duration of the game:

$T=2$	$T=3$	$T=4$	$T=5$	$T=7$	$T=9$	$T\to\infty$
$\varnothing$	$\varnothing$	$\varnothing$	$\rho \geq 0.89$	$\rho \geq 0.8$	$\rho \geq 0.78$	$\rho \geq 0.75$

The last table demonstrates that for $T = 2, 3, 4$, $\mathcal{P}_{\min}$ does not exist.

If condition (26) holds, then payment schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ can be constructed by Propositions 4 and 5. In $\mathcal{P}_1$, the payments for all non-terminal periods are the same as in $\mathcal{P}_{\min}$, and all savings from non-terminal periods are given to the players at the terminal stage. So, we need to define the payment at the terminal stage, that is,

$$p_i'(T) = p_i(T) + \sigma_i(T) = b + \sigma_i(T),$$

where

$$\begin{aligned}\sigma_1(T) + \sigma_2(T) &= \sum_{\tau=1}^{T-1} \frac{1}{\rho^{T-\tau}}2(a-(1-\rho)b-\rho d) + 2(a-b) \\ &= \frac{2}{\rho^{T-1}(1-\rho)}\left((b-d)(\rho-\rho^T) - (b-a)(1-\rho^T)\right).\end{aligned}$$

As the game is symmetric, it makes sense to assume that

$$\sigma_1(T) = \sigma_2(T) = \frac{1}{\rho^{T-1}(1-\rho)}\left((b-d)(\rho-\rho^T) - (b-a)(1-\rho^T)\right).$$

Finally, the payments of $\mathcal{P}_1$ are

$$\mathcal{P}_1 \quad \begin{cases} \text{For } t = 1, \ldots, T-1, \\ p'_i(t) := (1-\rho)b + \rho d. \\ \\ \text{For } t = T, \\ p'_i(T) := b + \sigma_i(T), \\ \text{with } \sigma_i(T) \geq 0 \text{ for any } i = 1, 2, \\ \text{and } \sigma_1(T) + \sigma_2(T) = \frac{2}{\rho^{T-1}(1-\rho)} \left((b-d)(\rho - \rho^T) - (b-a)(1-\rho^T) \right). \end{cases}$$

The payments in scheme $\mathcal{P}_2$ are the following:

$$\mathcal{P}_2 \quad \begin{cases} \text{For } t = T: \\ p''_i(T) := b; \;\; \text{and } B(T) := 2(b-a); \\ \\ \text{For } t = T-1, \ldots, 2: \\ p''_i(t) = \begin{cases} (1-\rho)b + \rho d, & \text{if } \rho + \rho^2 + \ldots + \rho^{T-t} < \frac{b-a}{a-d}, \text{ and} \\ & B(t) = 2[b - a - (\rho + \rho^2 + \ldots + \rho^{T-t})(a-d)]; \\ (1-\rho)b + \rho d + \xi_i(t), & \text{if } \rho + \rho^2 + \ldots + \rho^{T-t} \geq \frac{b-a}{a-d}, \text{ and} \\ & B(t) = 0, \end{cases} \\ \text{where } \xi_i(t) \geq 0, \; i = 1, 2 \text{ s.t. } \xi_1(t) + \xi_2(t) = 2[a - b + \rho(b-d) - \rho B(t+1)]; \\ \\ \text{For } t = 1: \\ p''_i(1) = (1-\rho)b + \rho d + \xi_i(1), \\ \text{where } \xi_i(1) \geq 0, \; i = 1, 2 \text{ s.t. } \xi_1(1) + \xi_2(1) = 2[a - b + \rho(b-d) - \rho B(2)]. \end{cases}$$

We note that the computations for the second payment scheme are more complicated, but for both schemes the payments are well defined.

For the set of parameters $b = 15$, $a = 10$, $d = 4$, $T = 4$, if $\rho \geq 0.48$, then the minimal payment scheme exists. For $\rho = 0.9$, we calculate the payments in schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ and report them in Table 1. The first two rows correspond to scheme $\mathcal{P}_1$, where the first line represents the general form of the payments, and the second line gives the payments assuming symmetry of players and dividing equally the savings. We can notice that players gets 5.1 at the first three stages, then obtain all savings at the last stage, that is, a payment of 28.215 in $t = 4$. While in the second scheme (last two rows), at the first two stages they obtain the maximal payment, that is, 10 in symmetric case, and they start making savings at stage 3. So, a player gets 5.5 at $t = 3$, and finally obtain 15 at the last stage. Both payment schemes satisfy efficiency property, and in any scheme the total discounted sum of players' payoffs is 68.78.

Table 1: Payments in schemes $\mathcal{P}_1$ and $\mathcal{P}_2$.

	$t=1$	$t=2$	$t=3$	$t=4$
$\mathcal{P}_1$	5.1	5.1	5.1	$15+\sigma_i(4)$ $\sigma_1(4), \sigma_2(4) \geq 0$ $\sigma_1(4)+\sigma_2(4) = 26.43$
$\mathcal{P}_1$ (symmetric)	5.1	5.1	5.1	28.215
$\mathcal{P}_2$	$5.1+\xi_i(1)$ $\xi_1(1), \xi_2(1) \geq 0$ $\xi_1(1)+\xi_2(1) = 9.8$	$5.1+\xi_i(2)$ $\xi_1(2), \xi_2(2) \geq 0$ $\xi_1(2)+\xi_2(2) = 9.8$	$5.1+\xi_i(3)$ $\xi_1(3), \xi_2(3) \geq 0$ $\xi_1(3)+\xi_2(3) = 0.8$	15
$\mathcal{P}_2$ (symmetric)	10	10	5.5	15

5 Link to cooperative games

We discussed in the introduction the differences between our payment schemes and an imputation distribution procedure. Here, we make a link between our payment schemes $\mathcal{P}_1$, $\mathcal{P}_2$ and an imputation of a cooperative game (M, v), where $v(K) : 2^M \to \mathbb{R}$ is the characteristic function satisfying $v(\varnothing) = 0$, and $K \subseteq M$. The value that any coalition K can achieve depends on the behavior of the players outside the coalition. Here, we retain the so-called γ characteristic function (CF), that is, $v(K)$ is the Nash equilibrium outcome of K in the noncooperative game between coalition K (acting as one player) and the left-out players (i.e., players in $M \backslash K$) acting individually. Finally, we recall that the set of imputations Y (or allocations) is defined by

$$Y = \left\{ (y_1, \ldots, y_m) \mid y_i \geq v(\{i\}), \forall i \text{ and } \sum_{i=1}^{m} y_i = v(M) \right\}.$$

So, for a vector $y = (y_1, \ldots, y_m)$ to be an imputation, it must be (i) individually rational, i.e., each player must, under cooperation, receive at least what she can get by acting alone, i.e., $y_i \geq v(\{i\}), \forall i$; and (ii) efficient, i.e., the grand coalition's total payoff $v(M)$ must be fully allocated.

Denote by $y(0, x_0)$ an imputation in the game starting at $t = 0$ with an initial state value x_0. Similarly, let $v(0, x_0, \{i\})$ be the γ-CF value of Player i in the game.

Proposition 6. *Consider a payment scheme $\mathcal{P}_1 = (p_i'(t) : i \in M, t \in \mathbb{T})$ satisfying the Feasibility, DIR, SAID, and Efficiency properties defined in Proposition 4. Then, the vector $y(0, x_0) \in \mathbb{R}^m$ defined by*

$$y_i(0, x_0) = \sum_{\tau=0}^{T} \rho^\tau p_i'(\tau), \; i = 1, \ldots, m, \tag{27}$$

is an imputation of the γ-CF cooperative game starting at time 0 with initial state x_0.

Proof. To show that $y(0, x_0)$ is an imputation in the game, we need to prove that it is individually rational, that is,

$$y_i(0, x_0) \geq v(0, x_0, \{i\}).$$

Adopting the γ-CF, we need to find the Nash equilibrium in the game when players of coalition K act as a single player and other players act as singletons. Therefore, $v(0, x_0, \{i\}) = J_i^{nc}(0, x_0)$. As DIR property is satisfied for $\mathcal{P}_1$, then

$$y_i(0, x_0) = \sum_{\tau=0}^{T} \rho^\tau p_i'(\tau) \geq J_i^{nc}(0, x_0) = v(0, x_0, \{i\}).$$

The second condition for the imputation to be satisfied is efficiency, i.e.,

$$\sum_{i \in M} \xi_i(0, x_0) = v(0, x_0, M),$$

which follows from the Efficiency property of $\mathcal{P}_1$. □

Remark 4. We note that the imputation defined by (27) may coincide with a specific cooperative solution, e.g., the equal surplus division, the Shapley value, or belongs to the γ-core. Finding a one-to-one correspondence between our schemes and the solutions of a cooperative game is not our focus. Still, the fact that both payment schemes define the vectors of total payoffs that are the imputations of a corresponding cooperative game makes them even more attractive.

Proposition 7. *Consider a payment scheme $\mathcal{P}_2 = (p_i''(t) : i \in M, t \in \mathbb{T})$ satisfying the Feasibility, DIR, SAID, Efficiency, and MRS properties defined in Proposition 5. Then, the vector $y(0, x_0) \in \mathbb{R}^m$ given by*

$$y_i(0, x_0) = \sum_{\tau=0}^{T} \rho^\tau p_i''(\tau), \; i = 1, \ldots, m, \tag{28}$$

is an imputation of the γ-CF cooperative game starting at time 0 with initial state x_0.

Proof. The proof is similar to the proof of Proposition 6, and the individual rationality of $y(0, x_0)$ follows from the DIR property satisfied for $\mathcal{P}_2$. The Efficiency of the payment scheme $\mathcal{P}_2$ implies the efficiency of the imputation $y(0, x_0)$. □

6 Example

We illustrate our results with an example of transboundary pollution control. The model is a discrete-time version of the infinite horizon game in Van der Ploeg and de Zeeuw (1992).[2]

Denote by $M = \{1, \dots, m\}$ the set of players representing countries, and by $\mathbb{T} = \{0, \dots, T\}$ the set of time periods. Production activities in each country generate revenues and, as a by-product, pollutant emissions, e.g., CO_2. Denote by $Q_i(t)$ Player i's production of goods and services at time $t \in \mathbb{T}$, and by $u_i(t)$ the resulting emissions at time $t \in \mathbb{T} \setminus \{T\}$, with $u_i(t) = h_i(Q_i(t))$, where $h_i(\cdot)$ is an increasing function satisfying $h_i(0) = 0$. Assuming a monotone increasing relationship between production and revenues, we can express the revenues, denoted by R_i, directly as a function of emissions. We adopt the following specification:

$$R_i(u_i(t)) = \alpha_i u_i(t) - \frac{1}{2}\beta_i (u_i(t))^2, \tag{29}$$

where α_i is a positive parameter for any $i \in M$.

Denote by $u(t) = (u_1(t), \dots, u_m(t))$ the profile of countries' emissions at time $t \in \mathbb{T} \setminus \{T\}$, and by $x(t)$ the stock of pollution at this time. The evolution of this stock is governed by the following difference equation:

$$x(t+1) = (1-\delta)x(t) + \sum_{i \in M} u_i(t), \quad t = 0, \dots, T-1, \tag{30}$$

$$x(0) = x_0, \tag{31}$$

where the initial state x_0 is given, and $\delta \in (0, 1)$ is a rate of pollution absorption by Mother Nature.

Each country suffers an environmental damage cost due to pollution accumulation. We assume that this cost is an increasing, convex function in this stock and retain the quadratic form $D_i(x(t)) = c_i(x(t))^2$, $i \in M$, where c_i is a strictly positive parameter.

Denote by $\rho \in (0, 1)$ the discount factor. Assume that Player $i \in M$ maximizes the following objective functional:

$$J_i(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \rho^t \left(R_i(u_i(t)) - D_i(x(t))\right) + \rho^T \Phi_i(x(T)),$$

subject to (30), and $u_i(t) \geq 0$ for all $i \in M$ and any $t \in \mathbb{T} \setminus \{T\}$, where $\mathbf{x} = \{x(t) : t \in \mathbb{T}\}$ is a state trajectory.

Let the payoff function of Player i at the terminal time T

$$\Phi_i(x(T)) = -d_i x(T).$$

Substituting $\Phi_i(x(T))$, $R_i(u(t))$, and $D_i(x(t))$ by their values, we get

$$J_i(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \rho^t \left(\alpha_i u_i(t) - \frac{1}{2}\beta_i (u_i(t))^2 - c_i(x(t))^2\right) - \rho^T d_i x(T).$$

[2] For a background on this class of games, see the survey in Jørgensen et al. (2010).

We use the following parameter values in the numerical illustration:

$$
\begin{array}{lll}
M = \{1,2,3\}, & \mathbb{T} = \{0,1,\dots,5\}, & \\
x_0 = 0, & \rho = 0.9, & \delta = 0.4, \\
\alpha_1 = 30, & \alpha_2 = 29, & \alpha_3 = 28, \\
\beta_1 = 3, & \beta_2 = 4, & \beta_3 = 4.5, \\
c_1 = 0.15, & c_2 = 0.10, & c_3 = 0.05, \\
d_1 = 0.15, & d_2 = 0.10, & d_3 = 0.05
\end{array}
$$

We consider the game with a feedback information structure, assuming the linear form of strategies and linear-quadratic form of the value functions in a state variable. Solving the joint maximization problem, we obtain the control (emissions) trajectories and the corresponding state values given in Table 2. The total payoff of the grand coalition in the game is 747.492.

Table 2: Cooperative control trajectories $u_i^*(t)$, $i \in M$, $t \in \mathbb{T} \setminus \{T\}$, and cooperative state trajectory $x^*(t)$, $t \in \mathbb{T}$.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$u_1^*(t)$	4.8191	4.0563	4.2195	5.5987	9.9100	
$u_2^*(t)$	3.3643	2.7922	2.9146	3.9490	7.1825	
$u_3^*(t)$	2.7683	2.2597	2.3685	3.2880	6.1622	
$x^*(t)$	0	10.9517	15.6792	18.9101	24.1818	37.7638

First, we define the minimal payment scheme. Recall that the payments in this scheme are given by (18)–(20). Next, we verify the feasibility condition given by (8) and (9). Table 3 contains the values used for determining the minimal payment scheme. In the first three rows one can find the

Table 3: The computations for the numerical example.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$BR_1(t,x_t^*)$	214.603	175.072	156.162	125.418	57.179	-5.665
$BR_2(t,x_t^*)$	188.254	150.199	125.728	95.755	43.245	-3.776
$BR_3(t,x_t^*)$	241.121	197.081	157.577	113.599	56.171	-1.888
$J_1^{nc}(t,x_t^*)$	163.692	110.159	96.975	92.719	57.166	-5.665
$J_2^{nc}(t,x_t^*)$	160.828	113.624	90.737	75.289	43.237	-3.776
$J_3^{nc}(t,x_t^*)$	229.723	181.512	142.274	104.325	56.167	-1.888
$p_1(t)$	57.038	34.526	43.286	73.957	62.277	-5.665
$p_2(t)$	53.075	37.044	39.549	56.834	46.644	-3.776
$p_3(t)$	63.748	55.262	55.338	63.045	57.871	-1.888
$\sum_{i\in M} p_i(t)$	173.861	126.832	138.173	193.835	166.792	-11.329
$\sum_{i\in M} \phi_i(t,x^*(t),u^*(t))$	244.934	178.190	147.357	164.736	166.779	
$\sum_{i\in M} \Phi_i(x^*(T))$						-11.329
$\sum_{i\in M} (\phi_i(t,x^*(t),u^*(t)) - p_i(t))$	71.073	51.358	9.184	-29.099	-0.013	
$\sum_{i\in M} (\Phi_i(x^*(T)) - p_i(T))$						0.000
Savings by t	71.073	130.328	153.993	142.005	157.770	175.300

payoffs that players can obtain by individually deviating from the cooperative state trajectory, i.e., $BR_1(t,x^*(t))$, $BR_2(t,x^*(t))$, $BR_3(t,x^*(t))$ used in the SAID property. The next three rows give the Nash equilibrium outcomes for subgames starting from cooperative states $x^*(t)$, i.e., $J_1^{nc}(t,x^*(t))$, $J_2^{nc}(t,x^*(t))$, $J_3^{nc}(t,x^*(t))$. They are used in the DIR property. We should notice that the best reply payoffs are larger than the ones in the Nash equilibria, $BR_i(t,x^*(t)) \geq J_i^{nc}(t,x^*(t))$ for any $i \in M$

and any $t \in \mathbb{T}$. Therefore, the definition of the minimal payment scheme $p_i(t)$ by equations (18)–(20) results in the following:

$$p_i(t) := BR_i(t, x^*(t)) - \sum_{\tau=t+1}^{T} \rho^{\tau-t} p_i(\tau),$$

with boundary condition

$$p_i(T) := \Phi_i(x^*(T)).$$

The components of the minimal payment scheme $p_i(t)$ are given in Table 3 for any $t \in \mathbb{T}$ and $i \in M$. We can easily compare the total payments to the players $\sum_{i \in M} p_i(t)$ with the total payoffs they earn at time $t \in \mathbb{T} \setminus \{T\}$, that is, $\sum_{i \in M} \phi_i(t, x^*(t), u^*(t))$ (the differences $\sum_{i \in M} (\phi_i(t, x^*(t), u^*(t)) - p_i(t))$ are also presented in Table 3). We conclude that, at the beginning of the game ($t = 0, 1, 2$) the total players' payoff is larger than the total payments in the minimal scheme, so there is no deficit. But at time $t = 3, 4$ the situation is the opposite (the differences are -29.099 and -0.013 for $t = 3$ and $t = 4$, respectively), so the players need to borrow money from the previous periods. The savings by period t given by

$$\sum_{\tau=0}^{t} \frac{1}{\rho^{t-\tau}} \sum_{i \in N} (\phi_i(t, x^*(t), u^*(t)) - p_i(t)),$$

are represented in the last row of Table 3. We notice that, for any t, these savings are positive, which means that the feasibility conditions (8) and (9) are satisfied and that the minimal payment scheme $\mathcal{P}_{\min}$ defined by $(p_i(t) : i \in N, t \in \mathbb{T})$ exists.

Now, we present two payment schemes defined in Propositions 4 and 5. According to $\mathcal{P}_1$, the payments are defined by (21) and organized in such a way that, until the terminal time, the payments are equal to the corresponding payments in the minimal scheme $\mathcal{P}_{\min}$, but at the terminal time, all savings (here of 175.300) are allocated to the players at time $t = 5$. The first payment scheme is defined in Table 4. By Proposition 4, the first payment scheme satisfies the Feasibility, DIR, SAID, and Efficiency properties. The payment scheme defined in Table 4 is nonunique because of the nonuniqueness in defining $\sigma_1(5), \sigma_2(5), \sigma_3(5)$ given in the last column of the table.

Table 4: The first payment scheme.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$p'_1(t)$	57.038	34.526	43.286	73.957	62.277	$-5.665 + \sigma_1(5)$
$p'_2(t)$	53.075	37.044	39.549	56.834	46.644	$-3.776 + \sigma_2(5)$
$p'_3(t)$	63.748	55.262	55.338	63.045	57.871	$-1.888 + \sigma_3(5)$
Conditions						$\sum_{i \in M} \sigma_i(5) = 175.300$, $\sigma_i(5) \geq 0$, $\forall i \in M$

The second payment scheme is introduced in Proposition 5. Its definition requires the computation of the amounts borrowed $B(t)$ from the previous savings to make the payments at time t. The values of $B(t)$ are given in Table 5. At time periods when $B(t) = 0$, the players do not set anything aside for future payments, i.e., the condition

$$\sum_{i \in M} (\phi_i(t, x^*(t), u^*(t)) - p_i(t)) - \rho B(t+1) \geq 0,$$

is satisfied and the players redistribute the amount

$$\sum_{i \in M} \xi_i(t) = \sum_{i \in M} (\phi_i(t, x^*(t), u^*(t)) - p_i(t)) - \rho B(t+1),$$

at time t if it is positive. The values

$$\sum_{i\in M}\xi_i(t)=\left[\sum_{i\in M}\left(\phi_i(t,x^*(t),u^*(t))-p_i(t)\right)-\rho B(t+1)\right]^+$$

are given in Table 5.

Table 5: The second payment scheme.

t	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$B(t)$	0	0	17.016	29.111	0.013	0
$\sum_{i\in M}\xi_i(t)$	71.073	36.044	0	0	0	0
$p_1''(t)$	$57.038+\xi_1(0)$	$34.526+\xi_1(1)$	43.286	73.957	62.277	-5.665
$p_2''(t)$	$53.075+\xi_2(0)$	$37.044+\xi_2(1)$	39.549	56.834	46.644	-3.776
$p_3''(t)$	$63.748+\xi_3(0)$	$55.262+\xi_3(1)$	55.338	63.045	57.871	-1.888
Conditions	$\sum_{i\in M}\xi_i(0)=71.073$, $\xi_i(0)\geq 0,\ \forall i\in M$	$\sum_{i\in M}\xi_i(1)=36.044$, $\xi_i(1)\geq 0,\ \forall i\in M$				

By Proposition 5, the second payment scheme satisfies the Feasibility, DIR, SAID, Efficiency, and MRS properties. The payment scheme defined in Table 5 is nonunique because of the nonuniqueness in defining $\xi_i(0)$, $i\in M$ and $\xi_i(1)$, $i\in M$ given in Table 5. We show that the minimal required savings (MRS) property is satisfied for the second payment scheme. $\mathcal{P}_2$ satisfies the property of minimal required savings if condition (15) is true for all $t\in\mathbb{T}\setminus\{T\}$, that is,

$$\begin{aligned}\sum_{\tau=0}^{t}\frac{1}{\rho^{t-\tau}}\sum_{i\in M}\Big(\phi_i\left(\tau,x^*(\tau),u^*(\tau)\right)-p_i''(\tau)\Big)=&-\sum_{\tau=t+1}^{T-1}\rho^{\tau-t}\sum_{i\in M}\Big(\phi_i\left(\tau,x^*(\tau),u^*(\tau)\right)-p_i''(\tau)\Big)\\&-\rho^{T-t}\sum_{i\in M}\Big(\Phi_i\left(x^*(T)\right)-p_i''(T)\Big)\end{aligned}$$

for payment scheme $\mathcal{P}_2$ defined by $p_i''(t)$.

Table 6 provides the values in the LHS and RHS of (15) for all $t\in\mathbb{T}\setminus\{T\}$. We can easily see from Table 6 that the values in the LHS and RHS are equal for any $t\in\mathbb{T}\setminus\{T\}$.

Table 6: Verification of the MRS property for the second payment scheme $\mathcal{P}_2$.

t	LHS	RHS
0	0	0
1	15.314	15.314
2	26.200	26.200
3	0.012	0.012
4	0	0

7 Existence of the minimal payment scheme

It is not possible to derive workable conditions for the existence of $\mathcal{P}_{\min}$ without referring to the data of the game, e.g., the functional forms, the duration and other parameter values. In this section, we provide some hints regarding the issue of existence of $\mathcal{P}_{\min}$.

For the repeated PD game in Section 4, we derive sufficient conditions under which $\mathcal{P}_{\min}$ exists. The results allow for the following two observations:

1. The duration of the game should be sufficiently long for the minimal payment scheme $\mathcal{P}_{\min}$ to exist. Further, the longer this duration, the larger is the interval of the discount factor under which $\mathcal{P}_{\min}$ exists. For one set of parameters and short game duration (2, 3, or 4 periods), $\mathcal{P}_{\min}$ does not exist for any discount factor values.
2. For a given game duration, $\mathcal{P}_{\min}$ exists only if the discount factor exceeds a certain threshold value. The larger the discount factor, the lower is this threshold and the easier is to satisfy the conditions for existence of $\mathcal{P}_{\min}$. This observation is in line with the results on existence of a subgame perfect equilibrium in infinitely repeated games.

Now, we discuss the existence conditions of $\mathcal{P}_{\min}$ for the game analyzed in Section 6. To illustrate, let us reconsider this game retaining the same parameter values, but letting $T = 2$, that is, $\mathbb{T} = \{0, 1, 2\}$. The control (emissions) trajectories and the corresponding state values are given in Table 7. The total payoff of the grand coalition in the game is 538.53. To check for the existence of $\mathcal{P}_{\min}$, we make the

Table 7: Cooperative control trajectories $u_i^*(t)$, $i \in M$, $t \in \mathbb{T} \setminus \{T\}$, and cooperative state trajectory $x^*(t)$, $t \in \mathbb{T}$.

	$t=0$	$t=1$	$t=2$
$u_1^*(t)$	7.02188	9.91	
$u_2^*(t)$	5.01641	7.1825	
$u_3^*(t)$	4.23681	6.16222	
$x^*(t)$	0	16.2751	33.0198

required calculations and report them in Table 8. As we can see, the Feasibility property does not hold. In particular, for $t = 0$, the feasibility condition

$$\sum_{i \in M} (\phi_i(0, x_0, u^*(0)) - p_i(0)) \geq 0,$$

is not satisfied as the LHS expression is equal to -13.311. The conclusion is that the planning horizon is too short to make the necessary savings to support cooperation, i.e., to satisfy the SAID property. Similar computations lead to the conclusion that $\mathcal{P}_{\min}$ does not exit for $T = 3$, but it exists for any $T \geq 4$. This lower bound to have the existence of $\mathcal{P}_{\min}$ provides a clear guide to the design of a cooperative contract. This observation is similar to the one made regarding the repeated PD game.

Table 8: The computations for the numerical example.

	$t=0$	$t=1$	$t=2$
$BR_1(t, x_t^*)$	234.957	105.801	−4.9530
$BR_2(t, x_t^*)$	167.477	75.660	−3.3020
$BR_3(t, x_t^*)$	149.418	72.379	−1.6510
$J_1^{nc}(t, x_t^*)$	219.986	105.789	−4.9530
$J_2^{nc}(t, x_t^*)$	158.107	75.652	−3.3020
$J_3^{nc}(t, x_t^*)$	145.172	72.375	−1.6510
$p_1(t)$	139.736	110.259	−4.9530
$p_2(t)$	99.383	78.632	−3.3020
$p_3(t)$	84.277	73.865	−1.6510
$\sum_{i\in M} p_i(t)$	323.396	262.755	138.173
$\sum_{i\in M} \phi_i(t, x^*(t), u^*(t))$	310.085	262.743	
$\sum_{i\in M} \Phi_i(x^*(T))$			-9.90594
$\sum_{i\in M} (\phi_i(t, x^*(t), u^*(t)) - p_i(t))$	−13.311	−0.012	
$\sum_{i\in M} (\Phi_i(x^*(T)) - p_i(T))$			0

8 Concluding remarks

In this paper, we proposed two payment schemes that ensure the sustainability of cooperation in dynamic games. We reiterate that their definitions do not require any particular assumption on the structure of the game, that is, on the functional forms of the payoff functionals and state dynamics.

Our contribution is a starting point for further developments on the design of payment schemes to sustain cooperation in dynamic games. The following developments are clearly of interest:

1. The design of similar payment schemes to sustain cooperation in differential games and in multistage games, including discrete-time stochastic games and dynamic games played over event trees.
2. The idea of credibility of punishing a deviator can be used to define new properties of the payment schemes. That is, the deviator is punished only if it is in the best interest of the other players to do so.
3. One may verify which properties from $P1$–$P6$ are satisfied for the payment schemes $\mathcal{P}_1$ and $\mathcal{P}_2$ when the utilities do explicitly depend on time, that is, they are given by $v_i(\phi_i(t, x^*(t), u^*(t)))$ for all $t \in \mathbb{T} = \{0, 1, \ldots, T-1\}$ and $v_i(\Phi_i(T, x^*(T)))$, for $i \in M$.
4. It is interesting to investigate how the ideas pursued here can be extended to infinite-horizon dynamic games. Intuitively, the longer the horizon, the easier to sustain cooperation. However, the absence of a terminal condition leads to a conceptual difficulty in defining the terminal payments.

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