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Short-term hydropower optimization in the day-ahead market using a nonlinear stochastic programming model

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Abstract : Hydropower producers participate in the electricity market by providing bids in the day-ahead market auctions. Making good bids that obey all market rules and consider uncertain prices for large, interconnected hydropower watercourses is challenging. This investigation aims to find bidding strategies that attend to the market aspects and all constraints relevant to short-term hydropower production. This paper presents a stochastic mixed-integer nonlinear model and a nonlinear heuristic method for the bidding optimization problem and shows a comparison of the model's performance in two case studies. The comparison of the two models shows that their results are close and that the heuristic method can reach the optimal solution after a few iterations. The numerical experiments are also compared with results from the Short-term Hydro Optimization Program (SHOP), which is a software used for operational planning in the Nordic electricity market.

Keywords: Bidding, Short-term hhydropower optimization, nonlinear programming, day-ahead market

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1 Introduction

Hydropower is one of the largest renewable energy sources and is one of the cheapest, cleanest, and most reliable sources of electricity production [1, 2]. Hydropower optimization is a nonlinear and nonconvex problem. The large size of the system and the uncertainty of some important input parameters have made managing this system very complex [3, 4]. The deregulation of the electricity market and increased competition has led to the development of decision-making tools in the electricity industry [5]. In addition, due to maximizing profits in the electricity market and creating a balance between current and future profits, attending to uncertain parameters such as prices is necessary [6, 7]. The day-ahead market is an essential part of the electricity market because it has the largest share of the market [8]. Due to the nature of this problem, the focus of this paper is to find the optimal strategy for day-ahead market price-taker hydropower producers by using a mixed-integer nonlinear stochastic model and a heuristic method. The models with integer variables are too computationally demanding; therefore, a heuristic method also is used for this problem.

There is extensive research about optimal bidding; however, in the existing literature, the main focus is thermal generation. Hydropower plants usually have low start-up costs and are flexible in high ramping. For hydropower producers, optimal scheduling of water reservoir volume and releasing water is very important. Due to the expansion of wholesale electricity markets around the world, the importance of determining the optimal bidding in hydropower systems is increasing [9]. A multi-stage stochastic Mixed-Integer Linear Programming (MILP) model to optimize bids for 2 hours ahead market in Canada is presented in [10]. Operational constraints in the cascaded rivers and uncertain inflows and prices are considered in the model. One of the essential research on optimizing bidding is presented by Fleten et al. in the NordPool system [11]. The authors proposed a two-stage MILP in the day-ahead markets with uncertain prices. This model provides a price-dependent bidding curve, and the Nordic market rules and unit commitment decisions are included. In hydropower production aspects, the net water head is ignored. This model is extended in [12], and the coordinated bidding optimizing model is proposed by adding the intraday market. Another extension work of [11] is the multi-stage approach presented in [13], which considers the bid decisions in the day-ahead and intraday markets by integrating the short-term intraday with long-term inter-day decisions. The problem is defined as a Markov decision process and is solved by using approximate dual dynamic programming. To evaluate the stochastic bidding model in the interconnected river systems in [8], a stochastic MILP model without considering the effect of the water head is presented. In this model, both the bidding problem and the actual operational dispatch are modelled, and prices and inflows are considered uncertain. Uncertainty increases the complexity of the problem, and it may be too difficult to find a solution in MILP models, especially in large interconnected river systems; therefore, a stochastic linear model is formulated instead of the MINLP model in [14]. In this study, the effect of the linear approximation of start-ups on the quality of the results and the solution time is investigated. Another solution method in the unit commitment problem that the power producers in the Nordic market have widely used is Successive Linear Programming (SLP). In [15], an SLP model for operational stochastic short-term hydropower with uncertain future prices and inflow for the Nordic power industry is presented. This method has been implemented in SHOP, and the nonlinear head effect is modelled. In [16], the stochastic SLP model is used to optimize the power production in the hydropower bidding problem; a greedy algorithm is also presented to reduce the bid matrix. The water released is linked to power production by a piecewise linear concave production function for each generator. This study has shown that the problem size and computational time grow with the number of scenarios. The deterministic and stochastic models to obtain a bidding strategy are compared in [17]. Based on the results, the stochastic method has a better outcome for participants in the day-ahead market than deterministic models, as found in [8, 10, 11], and [15].

Nonlinear and nonconvex relationships are between decision variables such as water height, water discharge and production efficiency in the hydropower problem. Dynamic programming can handle nonlinearity and nonconvexity, but in large problems, finding the solution is hard, which is called "the curse of dimensionality.” Therefore, in most hydropower optimization problems, either the nonlinear effects are ignored or linear approximation methods reported in the literature [9, 18].

SHOP is an optimization model for scheduling hydroelectric power plants for daily operations that is provided as a software by SINTEF Energy Research. The model can handle various operational, physical and market constraints in complex hydro systems [19, 20]. The solution process in SHOP consists of two parts: unit commitment and unit load dispatch. The decision of turbines on /off in each period is determined by the unit commitment problem. The process is that the MILP problem with an estimation of the reservoir trajectories is solved. Then a number of iterations are performed to stabilize the head changes. The volume and water head of the reservoir are updated after each iteration, and this process continues until the stop conditions are met, and the unit commitment problem is solved. In the second part, the linear problem, unit load dispatch, is activated. The binary variables obtained in the unit commitment problem are fixed, and an LP model is used to obtain exact generation [21].

As reflected in the literature, mixed integer models have been widely used in the bidding optimization problem. In often studies, linearization and approximation techniques have been used to solve the problem. This paper presents a two-stage Mixed-Integer Nonlinear Programming (MINLP) for the price-taker producers in the day-ahead bidding based on the Nordic market. Instead of linearization and discretization, the maximum power output surface of the water discharge and volume of the reservoir for each turbine combination is used, leading to a nonlinear representation of the functions. This model considers the operational constraints of power production in the hydropower system and market rules. However, the existence of integer variables and their combination with stochastic programming raises the complexity of the problem. The contribution of this article is that an MINLP model is introduced to optimize bidding for the day-ahead market. Also, a heuristic method to solve the hydropower stochastic MINLP bidding problem by solving problems with less complexity in an iterative process is presented. The heuristic method can reach the right result in a short time. The two methods, more precisely an exact MINLP and the heuristic MINLP, are compared on datasets from the Nordic market and show that the average income in the MINLP model is slightly better in the examined cases, but the average solution time in the heuristic method is shorter. In some instances, the MINLP model could not find a solution.

This paper is organized as follows: The MINLP model and the bidding structure in the day-ahead market are described in Section 2. In Section 3, a two-stage mixed integer nonlinear stochastic model, as well as an iterative heuristic method for solving the two-stage nonlinear bidding problem, are presented. In Section 4, the results of the case studies are provided, and the result for two cases are compared with SHOP to evaluate the models, and the conclusions are presented in Section 5.

2 Short-term hydropower scheduling and bidding problem

Short-term hydropower scheduling prepares the optimal strategy for daily operational plans. In order to achieve the desired performance, producers are looking to maximize revenues or minimize costs [22]. There are several methods for short-term hydropower optimization problems that can be divided into two general categories: exact methods and heuristic methods [23]. MILP models [5, 24] are widely used because some binary decisions, such as start-ups/shutdown and active units, cause the model to have integer variables. In addition, the efficiency of the turbine depends on water discharge and water head, which makes the problem model nonlinear [9, 25]. Also, short-term planning can be modelled in terms of plant-based or unit-based. In the plant-based models, the problem is based on the aggregation of the plant level, and in the unit-based models, the operational and physical conditions like turbine efficiency and limitations of the dispatch of the unit are considered [18]. The combination of turbines in operation can be used, which causes that in addition to all the advantages of the unit-based conditions, it also reduces the complexity of the problem, like the model presented in [26]. In Section 2.1, the MINLP model based on the combination of turbines is presented in details.

To participate in the Nordic day-ahead market, hydropower producers offer their bid matrix, a table containing the prices and power volumes per hour for the coming day to the market operator. This bid volume for each hour should not decrease as the price increases [16]. These rules, as well as the relationship between the bid matrix and scenario for day-ahead prices and committed production, are presented in Section 2.2.

2.1 Nonlinear short-term hydropower optimization

As mentioned, hydropower optimization is a nonlinear problem generally, and it has operational constraints that should be considered in the model. This Section introduces the power production, the turbine's efficiency, the turbine's combination of turbines, and the operational constraints in the nonlinear model.

The power production in the hydro system depends on the water discharge and water head, and turbine efficiency [27]. Power output (kW), in a single turbine is given as

$$p(q,h) = g * \eta(q) * q * h(Q,v), \tag{1}$$

where p is the power output (kW), g is the gravitational acceleration (m/s^2), η is the turbine-generator efficiency, q is the turbine water discharge(m^3/s), Q is the total water discharge(m^3/s), and h is the net water head (m). The gross water head is calculated from the difference between the tailrace elevation and forebay elevation, and the water friction in the penstock causes the reduction in the water head, which is called penstock losses. Therefore, the net water head is calculated as shown in Equation (2):

$$h(Q,v) = fb(v) - tl(Q) - pl(Q,q), \tag{2}$$

where fb is the forebay level of the reservoir unit, and v is the volume of the reservoir (m^3), tl is the tailrace level of the reservoir unit, pl is the penstock losses of the unit (m). As mentioned, one of the most important factors in power production is turbine efficiency, and each turbine has its own efficiency. Turbine efficiency is important for power producers because the power production can be different for different units, even under similar conditions like the same net water head and water discharge, if the turbine efficiency is different. In addition to the total water discharge and water head, another factor that affects power generation in operational reality is the number of active turbines. Instead of working with turbines individually, the turbines combination can be used. For example, suppose a power plant has a total of 4 turbines. In that case, the number of possible combinations for 1 active turbine is 4 and 6 combinations for 2 turbines, 4 for 3 turbines, and 1 for 4 active turbines. In a hydropower system, if the number of turbines is large, the number of combinations will increase, making the problem more complex. Instead of working with all combinations, the maximum power output surface can be used for the number of turbines in each combination. The maximum output for each turbine combination is obtained by considering the water discharge and the volume of the reservoir.

The first part of the two-phase model presented in [28] is used to obtain optimal power production. A MINLP is introduced to determine the power output, water discharge, reservoir volume and the number of turbines in the first phase. The second phase is the unit commitment problem where the start-up costs are penalized. Operational constraints are considered in the first phase of the model, which is the loading problem. In the loading problem, the water discharge, q_t^c, and the volume of the reservoir, v_t^c, for each power plant C at the period T is limited by maximum and minimum levels, as in Equation (3) and (4).

$$q_{min}^c \leq q_t^c \leq q_{max}^c, \qquad \forall t \in T, c \in C. \tag{3}$$

$$v_{min}^c \leq v_t^c \leq v_{max}^c, \qquad \forall t \in T, c \in C. \tag{4}$$

The maximum power output surfaces are obtained by polynomial equations and are used in the power production function equation, so the nonlinear relationship between water discharge and the volume of the reservoir is considered. The power production equations are:

$$\chi^c_{j,t}(q^c_t, v^c_t) z^c_{j,t}, \qquad \forall t \in T, c \in C. \tag{5}$$

where $\chi^c_{j,t}(q^c_t, v^c_t)$ are the power output function for the surface j at period t, and plant c (MW), and binary variable $z^c_{j,t}$ has value 1 if the surface j is chosen at the period t. Equation (6) ensures that only one surface is selected at each hour from the time horizon.

$$\sum_{j \in J} z^c_{j,t} = 1, \qquad \forall t \in T, c \in C. \tag{6}$$

Water balance equations are given by:

$$v^c_{t+1} = v^c_t - \zeta w_t(q^c_t + g^c_t) + \zeta \delta_t + \sum_{r \in R} \zeta w_t(q^r_t + g^r_t), \qquad \forall t \in T, c \in C. \tag{7}$$

Equation (7) ensure the water balance of the power plants that are connected in series. The volume of the reservoir in the next hour, v^c_{t+1} is the volume of the reservoir at the period t, v^c_t minus the water discharge used for power production and spillage, $w_t(q^c_t + g^c_t)$, and ζ is the conversion factor from water discharge (m^3/s) to the reservoir volume (Mm^3/h), and plus inflow, δ_t, and water flow from upstream reservoirs into the reservoir, $w_t(q^r_t + g^r_t)$. The initial volume, v_1 and final volume, v_{final}, are considered input parameters to the model. For this problem in two phases, namely the loading and unit commitment problems, it was shown in [28] that integer variables could be relaxed, allowing to solve of a continuous nonlinear problem in practice. Therefore, the solution will still be the integer, although the z variables are continuous in the formulation.

2.2 Bid structure

Hydropower producers offer their bids for the following day to the market operator in the day-ahead market, which is the main part of the European electricity market because most exchanges are done in these auctions. Producers and consumers submit their offer to the operator of the market organizer in the day-ahead market before noon. The marked operator calculates bid prices after receiving bids and offers from all market participants and then announces publicly at around 1 p.m. each day. Committed power is calculated by linear interpolation between each producer's bid curve and market price by the market operator. There are other markets, such as intraday and balancing to cover obligations, for the delivery of physical power. These markets increase flexibility and system stability [9, 29]. The balancing or real-time market is where the transaction and bidding are done near the operating hour, about 45 minutes or earlier. This market is necessary for ramping flexibility, and it is organized by the Transmission System Operator (TSO) as a single buyer[9].

The power producer participating in the day-ahead market can submit the hourly bid. As shown in Figure 1 the hourly bid includes the volume power offered for hour t, $XD_{i,t}$, and the price, $P_{i,t}$. The number of bid points, $I = \{ 1, 2, ..., I\}$, are specified by market rules.

The user-determined set of fixed hourly prices, $P = \{ p_1, p_2, ..., p_I\}$, is used to avoid nonlinear relation to determining bid volume and prices [11]. The hourly market prices are denoted by ρ_t and committed power volume at hour t, YD_t, is calculated with linear interpolation between the bid points on a bidding curve in Equation (8).

$$YD_t = \frac{\rho_t - p_{i-1}}{p_i - p_{i-1}} XD_{i,t} + \frac{p_i - \rho_t}{p_i - p_{i-1}} XD_{i-1,t} \qquad if p_{i-1} \le \rho_t \le p_i. \tag{8}$$

In the Nordic market, the bid by increasing prices must be non-decreasing, as shown in Equation (9).

$$XD_{i,t} \ge XD_{i-1,t} \qquad t \in T, i \in I. \tag{9}$$

More details about the bidding curve and the problem of determining bids are presented in [11, 30, 31].

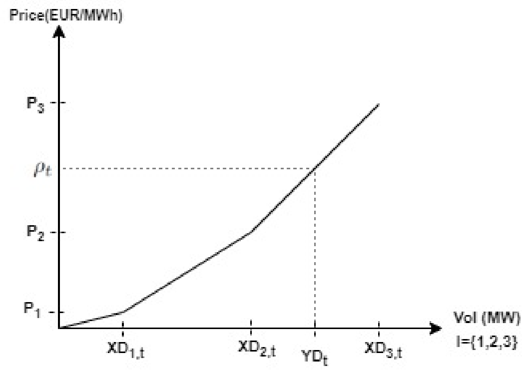

Figure 1: Example of an hourly binding curve

3 Methodology

In the day-ahead market, the prices are uncertain, so the model must consider this uncertainty to choose the optimal strategy. All the equations, including the production function and operating constraints, as well as the bidding structure presented in Section 2, are updated considering the uncertainty of the prices and a two-stage mixed integer nonlinear stochastic model is presented to optimize the bidding problem in Section 3.1.

Section 3.2 presents an iterative heuristic method to solve the bidding problem. The purpose of this method is to find a suitable solution with the number of iterations and some changes in the model and integer variables in a shorter time.

3.1 Two-stage mixed integer nonlinear stochastic model

This two-stage nonlinear stochastic model allows determining the bid volume in the first-stage decisions at hour t, $XD_{i,t}$. Second-stage decisions are committed hourly volumes, $YD_{s,t}$, and these decisions depend on the price scenarios ρ_t^s. Second-stage decisions are committed hourly volumes, $YD_{s,t}$, and these decisions depend on the price scenarios ρ_t^s and positive and negative imbalance between committed volume and power production are Zd and Zu. The objective function is to maximize the profit from offers, where π^s is the probability of each scenario, and α, β are the reward and penalty of participating in the balancing market. The MINLP stochastic model for the day-ahead market is given by:

$$\max \sum_{s \in S} \pi^s (\sum_{t \in T} \rho_t^s YD_t^s + \sum_{t \in T} (\alpha_t \rho_t^s Zu_t^s - \beta_t \rho_t^s Zd_t^s)) \tag{10}$$

Subject to:

$$YD_t^s = \frac{\rho_t^s - p_{i-1}}{p_i - p_{i-1}} XD_{i,t} + \frac{p_i - \rho_t^s}{p_i - p_{i-1}} XD_{i-1,t}, \qquad \forall t \in T, s \in S, i \in I, \tag{11}$$

$$XD_{i,t} \geq XD_{i-1,t}, \qquad \forall t \in T, i \in I, \tag{12}$$

$$H_t^s = \chi_{j,t}^s(q_t^s, v_t^s) z_{j,t}^s, \qquad \forall t \in T, j \in J, s \in S, \tag{13}$$

$$\sum_{j \in J} z_{j,t}^s = 1, \qquad \forall t \in T, s \in S, \tag{14}$$

$$v_{t+1}^s = v_t^s - \zeta w_t (q_t^s + g_t^s) + \zeta \delta_t + \sum_{r \in R} \zeta w_t (q_t^{s,r} + g_t^{s,r}), \qquad \forall t \in T, s \in S, \tag{15}$$

$$q_{min} \leq q_t^s \leq q_{max}, \qquad \forall t \in T, s \in S, \tag{16}$$

$$v_{min} \leq v_t^s \leq v_{max}, \qquad \forall t \in T, s \in S, \tag{17}$$
$$v_1 = v_{Initial} \tag{18}$$
$$v_T \geq v_{final} \tag{19}$$
$$YD_t^s - H_t^S = Zd_t^s - Zu_t^s, \qquad \forall t \in T, s \in S, \tag{20}$$
$$v_t^s \geq 0, q_t^s \geq 0, \qquad \forall t \in T, s \in S, \tag{21}$$
$$Zu_t^s \geq 0, Zd_t^s \geq 0, \qquad \forall t \in T, s \in S, \tag{22}$$
$$YD_t^s \geq 0, H_t^s \geq 0, \qquad \forall t \in T, s \in S, \tag{23}$$
$$v_t^s, q_t^s, Zu_t^s, Zd_t^s, YD_t^s, H_t^s \in R, \tag{24}$$
$$z_{j,t}^s \in B. \tag{25}$$

Constraints (11) are the piecewise linear interpolation of the offer curve, which is the actual dispatch in scenario S and at hour t. Constraints (12) ensure that as the price increases, the bid curve is non-decreasing. Equation (13) is the nonlinear production function for each price scenario at hour t, H_t^s, and is obtained from the MINLP model presented in Section 2.1. Constraints (14) limit the model to choose only one active turbine combination per hour t. The reservoir balance constraints are in Equation (15), constraints (16) are water discharge bounds and constraints (17) are the limitation of reservoir storage level, and (18)–(19) specify initial and final volumes. Constraints (20) are the imbalance between the committed volume and the power production per hour, the shortage is bought from the balancing market, and the extra energy is sold to the balancing market.

3.2 Nonlinear heuristic bidding

Using the maximum output surface instead of all possible turbine combinations reduces the complexity of the problem. For example, all possible combinations for four turbines become 24 integer variables. Instead of these 24 variables, we use four maximum output surfaces for each number of active turbines. Despite the significant reduction in the combination of turbines, it is still a very complicated and time-consuming task to check all the active turbines within a time horizon of 24 hours or more, as well as with many scenarios. Solving the MINLP problem is associated with challenges; the computation time usually significantly increases in large-scale and complex problems. Therefore, a heuristic method is proposed that can reach a suitable solution with a low number of iterations. The main problem is broken into smaller problems and solved by an iterative approach. The problem-solving steps are shown in Figure 2.

The loading problem is solved for all scenarios and problem inputs, including the initial and final volume and inflows, as shown in box 1. of Figure 2. Equation (26) is the objective function of the loading problem, which is to maximize the revenue in price scenarios. Equations (27)–(34) are the operational constraints of the hydropower problem.

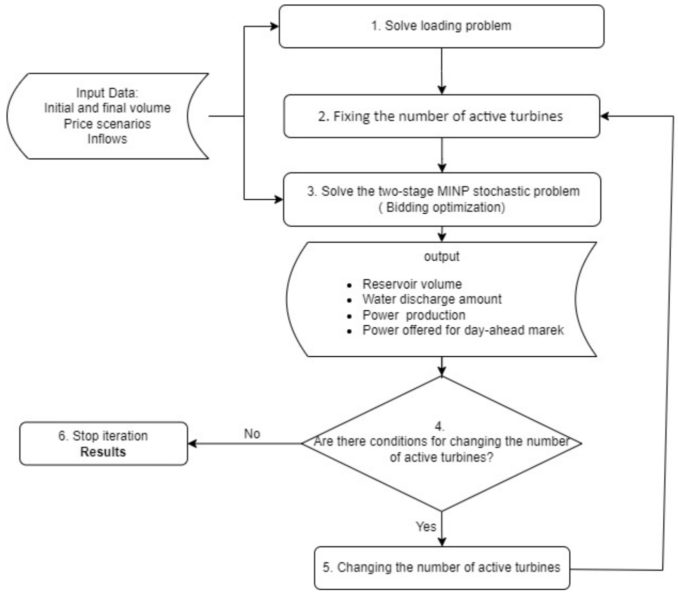

Figure 2: Flowchart of a heuristic method for the bidding optimization for the day-ahead market

$$\max \sum_{s\in S} \pi^s \sum_{t\in T} \sum_{j\in J} \rho_t^s \ \chi_{j,t}^s(q_t^s, v_t^s) z_{j,t}^s \tag{26}$$

Subject to:

$$v_{t+1}^s = v_t^s - \zeta w_t(q_t^s + g_t^s) + \zeta\delta_t + \sum_{r\in R} \zeta w_t(q_t^{s,r} + g_t^{s,r}), \qquad \forall t \in T, s \in S, \tag{27}$$

$$\sum_{j\in J} z_{j,t}^s = 1, \qquad \forall t \in T, s \in S, \tag{28}$$

$$v_0 = v_{Initial} \tag{29}$$

$$v_T \geq v_{final} \tag{30}$$

$$q_{min}^s \leq q_t^s \leq q_{max}^s, \qquad \forall t \in T, s \in S, \tag{31}$$

$$v_{min}^s \leq v_t^s \leq v_{max}^s, \qquad \forall t \in T, s \in S, \tag{32}$$

$$v_t^s \geq 0, q_t^s \geq 0, \qquad \forall t \in T, s \in S, \tag{33}$$

$$z_{j,t}^s \in B, \qquad \forall t \in T, \in J, s \in S. \tag{34}$$

Since the coefficients of the constraints (28)–(30) are 0 and 1 and have only one element, the total unimodularity conditions in [32] are satisfied, so we can relax the integer variable and solve this problem for all scenarios. After solving the first step, the number of active turbines and the optimal power production in each scenario are determined. Then, we fix the integer variable, the number of active turbines in each scenario. The MINLP stochastic problem, equations (10)–(25), is solved with a fixed integer variable, as shown in box 3. of Figure 2. Due to the absence of the integer variable in the nonlinear problem, the problem is solved as a continuous nonlinear problem. Fixing the integer variable may have introduced an error to the model, so the effect of this error can be reduced during an iterative process. For this purpose, it is reviewed whether the power production value will be increased by changing the number of active turbines, and the conditions mentioned in box 4. of Figure 2 are as follows. After solving the bidding problem, the amount of power generation, the reservoir volume, and the water discharge in each scenario are known. On the other hand, there are maximum power output surface equations for each turbine combination, as explained in Section 2.1. The results obtained from the bidding problem can be replaced in the nonlinear surface equations as input. If there is a number of activities turbines that provides better power production with the same input, the number of active turbines will be changed. Suppose there are three turbines in the hydro plant, so we have three maximum surface equations. Based on the solution of step one in scenario S, the two active turbines are selected at time t, and the third step is solved with a fixed integer value.

$$z^s_{j,t} = z^s_{2,t} = 1, \qquad \forall t \in T, s \in S \tag{35}$$

The results obtained in the third step include the power production, H^{*s}_t, water discharge, q^{*s}_t, and the reservoir volume v^{*s}_t at time T and scenario S.

$$H^{*s}_t = \chi^s_{2,t}(q^{*s}_t, v^{*s}_t) z^s_{2,t}, \qquad \forall t \in T, s \in S \tag{36}$$

In the third step, we put the obtained results in the equations of other turbine combinations, i.e. one active turbine,H^{1s}_t, and three active turbines, H^{3s}_t.

$$H^{1s}_t = \chi^s_{1,t}(q^{*s}_t, v^{*s}_t) z^s_{1,t}, \qquad \forall t \in T, s \in S \tag{37}$$

$$H^{3s}_t = \chi^s_{3,t}(q^{*s}_t, v^{*s}_t) z^s_{3,t}, \qquad \forall t \in T, s \in S \tag{38}$$

If the results were better, the turbine combination is changed.

$$H^{1s}_t(q^{*s}_t, v^{*s}_t) \geq H^{*s}_t(q^{*s}_t, v^{*s}_t) \geq H^{3s}_t(q^{*s}_t, v^{*s}_t), \qquad \forall t \in T, s \in S \tag{39}$$

It means that there is a number of active turbines that has a higher output with the same input than the previous condition. Therefore, instead of checking all turbine combinations, the number of active turbine changes only when the value of power production increases. Figure 3 shows the maximum output surface for three turbines, drawn in two dimensions for simplicity. As shown in the figure, after solving the stochastic model with a fixed integer variable of two turbines, the turbine combination will change in the next iteration if the amount of power production is in the marked parts. If the power production amount is in part A, the number of active turbines changes from two to one, and if it is in part B, the number of active turbines changes from two to three. In the next iteration, the number of changed turbines is fixed and the stochastic programming problem is resolved again. This iteration process continues until there is no change in the number of active turbines, after which the iteration stops.

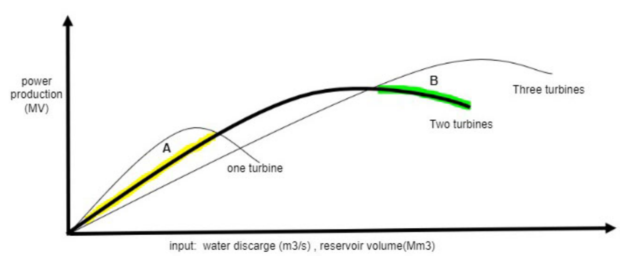

Figure 3: Maximum output surface for three turbines

4 Case study

The MINLP model and heuristic method presented in the previous section are tested on two cases with the data extracted from SHOP runs. This data includes power production values with different reservoir volumes and water discharge values for each turbine combination. Nonlinear equations of the maximum power output surface of each number of the active turbine are obtained using extracted data and polynomial approximation. The proposed method focuses on the day-ahead market, so the purchase amount in the balanced market is penalized, and selling the surplus amount is rewarded. Price scenarios, as shown in Figure 4, are selected to represent a typical pattern in the Nordic market, and inflows are regarded as deterministic. The summary of cases is shown in Table 1. All test cases are solved for 24 hours planning horizon.

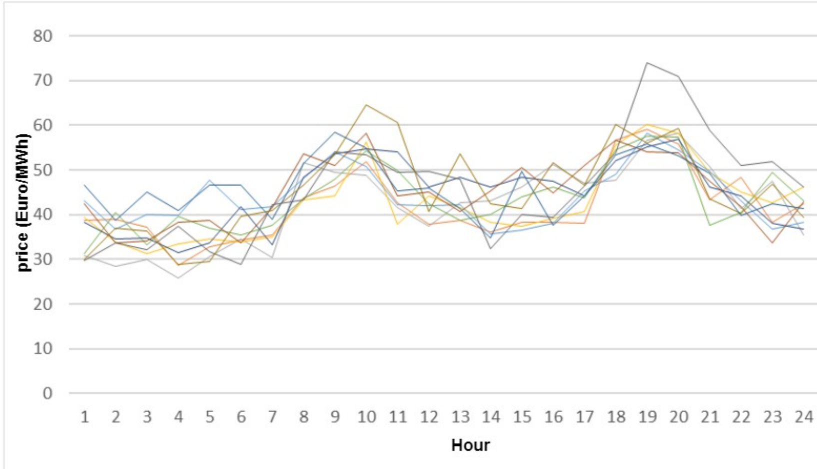

Figure 4: Stochastic prices

Table 1: Case study

Case study	Number of instances	Number of reservoirs	Number of turbines
Case A	20	1	4
Case B	60	2	6
Validation results	9	2	6

In each case, several instances with different input parameters, including the initial and final volume of the reservoir and the inflows, have been investigated. In each instance, to obtain the difference between the results of the objective function in the MINLP model and the heuristic method, the difference is defined as:

$$Difference(\%) = (1 - \frac{Objective\ function\ of\ Heuristic(EUR)}{Objective\ function\ of\ MINLP(EUR)}) \times 100 \quad (40)$$

To evaluate the presented methods, the results of these two models have been compared with SHOP. This program, based on successive linear programming and may include mixed integer programming, can solve problems with a large number of cascaded water courses. The improvement percentage in the objective function of the MINLP model and the heuristic method with SHOP is shown in Equations (41) and (42).

$$MINLP\ improvement(\%) = (1 - \frac{Objective\ function\ of\ SHOP\ (EUR)}{Objective\ function\ of\ MINLP\ (EUR)}) \times 100 \quad (41)$$

$$Heuristic\ improvement\ (\%) = (1 - \frac{Objective\ function\ of\ SHOP\ (EUR)}{Objective\ function\ of\ heuristic\ (EUR)}) \times 100 \quad (42)$$

BONMIN [33] is an effective and efficient open-source solver used to solve MINLP and non-convex problems based on Cbc [34] and Ipopt [35] as building blocks. The stochastic MINLP and heuristic model are implemented in Julia, and the optimization software to solve stochastic MINLP is the BONMIN, and for the stochastic heuristic model, the Ipopt is used. A laptop computer with an Intel Core i5 Processor and 8 GB of RAM is utilized to solve the models.

4.1 Case A

This case study uses a power plant with four turbines and a reservoir. The maximum power production capacity of this hydropower system with four turbines is 345 MWh, and the volume of the reservoir when full is 104.16 Mm^3. The total number of turbine combinations is 24, but for each number of active turbines, the maximum output surface is used, which is 4. A maximum output surface should be selected per hour, so 4 binary variables are considered in the MINLP model for each hour. 20 instances of various reservoir conditions, including full, half full, and almost empty, as well as 10 price scenarios and 24-hour time horizon, are examined. Inflows are considered deterministic. The comparison between the MINLP model and the heuristic method is shown in Table 2. As mentioned in Section 3.1, the value of the objective function is revenue in euros for participation in the day-ahead market and the balancing market, in case of an imbalance between the power offered volume and the committed volume. Equation (40) is used to calculate the difference percentage of the objective function of the two models. The number of active turbines selected in the first iteration will not change in the next iteration of the heuristic method if the initial and final volumes are such that 4 turbines, the maximum power production of the hydropower plant, are on for all scheduling time. To make the comparison between the MINLP model and the heuristic method more meaningful, the input parameters, such as the initial and final reservoir volume and the inflow, are set in such a way that there is not enough water to produce full in all hours.

Table 2: Comparison income of stochastic MINLP model and stochastic heuristic method results in case A

Instance	MINLP (EUR)	Heuristic (EUR)	Difference (%)	Time MINLP (s)	Time HEU (s)	Diff Time (s)	Number of iterations
1	296,076	296,052	0.00%	13.10	25.55	-12.45	2
2	186,579	186,512	0.04%	114.29	51.12	63.18	4
3	324,377	324,381	-0.00%	22.28	23.10	-0.83	3
4	234,886	234,877	0.00%	31.75	30.88	0.87	3
5	178,121	178,043	0.04%	50.78	39.91	10.87	4
6	255,380	255,341	0.02%	50.75	43.39	7.36	4
7	244,565	244,520	0.02%	114.73	42.47	72.26	4
8	360,184	360,170	0.00%	19.48	34.47	-14.99	3
9	136,660	136,689	-0.02%	18.65	25.05	-6.39	3
10	218,479	218,359	0.05%	28.81	42.28	-13.47	4
11	138,268	138,246	0.02 %	36.55	32.53	4.02	4
12	212,724	212,675	0.02%	17.81	51.37	-33.56	5
13	282,276	282,291	-0.01%	18.64	49.04	-30.40	4
14	276,439	276,446	-0.00%	25.36	43.81	-18.46	4
15	170,818	170,776	0.02 %	15.77	43.28	-27.51	4
16	240,744	240,743	0.00 %	132.10	36.28	95.83	3
17	221,723	221,677	0.02 %	79.46	46.63	32.83	4
18	241,412	241,375	0.02 %	99.89	33.82	66.07	3
19	219,511	219,492	0.01%	76.17	46.03	30.14	4
20	202,485	202,465	0.01 %	18.54	45.39	-26.85	4

The model results in Table 2 show that both the MINLP model and heuristic methods are efficient and can reach the solution in a short time. The average difference of the objective function value is 0.014% which is higher in the MINLP model. The average solution time for 20 instances in the MINLP model is 51.15 seconds, and in the heuristic method is 40.04 seconds. Although the solvers of the MINLP model and the heuristic method are different, the values of the objective functions are close, and there is not much difference between them. In the heuristic method, there is no guarantee of reaching the optimal solution, and because the solver of the two methods is not the same, some difference in the results is normal. Some hours of instance 2 are selected for comparison of the bid volume and bid curve. Figure 5 shows the bid volume of the MINLP model and the heuristic method in the day-ahead market for ten scenarios at 5, 11, 18 and 21 hours in instance 2 and Case A. In this figure, the amount of bid volume in the MINLP model and the heuristic method is presented in the specified hours for 10 price scenarios.

Figure 6 shows the bid curve at 5, 11, 18 and 21 hours for instance 2, case A. The bid curves in the MINLP model and the heuristic method have been compared in this figure. Usually, the bid curves are close at high prices for each hour in both models, and the differences are at low prices.

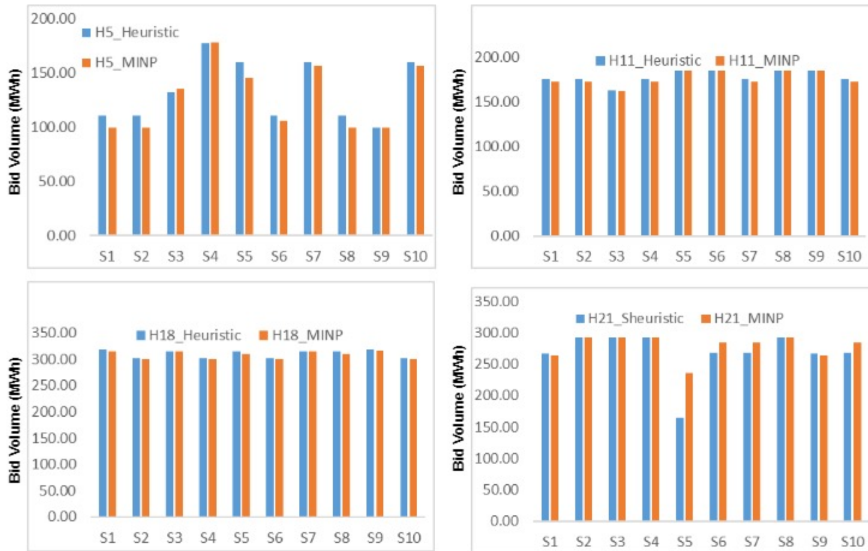

Figure 5: Comparison of the bid volume per stochastic price at 5, 11, 18 and 21 hours and different scenarios in instance 2, case A

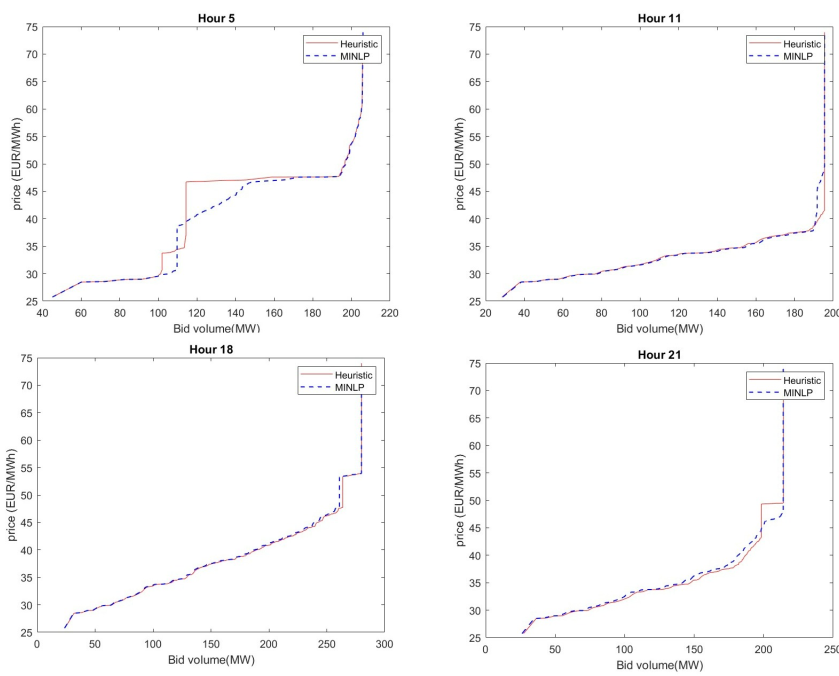

Figure 6: Comparison of the bid curve at 5, 11, 18 and 21 hours and different scenarios in instance 2, case A

4.2 Case B

In the second case study, two power plants are connected in series. The topology of the system is illustrated in Figure 7. The first power plant has two turbines, a maximum power generation capacity of 240 MWh, and a maximum reservoir volume of 41.66 Mm^3. The second power plant has four turbines, a maximum power generation capacity of 345 MWh, and a maximum reservoir volume of 104.16 Mm^3. The simulation is for a Norwegian system that participates in the day-ahead market with a 24-hour time horizon. The MINLP model and heuristic method were tested on 60 instances for a planning horizon of 24 hours, with the initial and final volumes different, reservoirs almost empty, half-full and full. The inflows are considered deterministic and from low to high inflows. As shown in Table 3, rows 1 to 20 show results for low inflow, 21 to 40 for medium inflow, and 41 to 60 for high inflow.

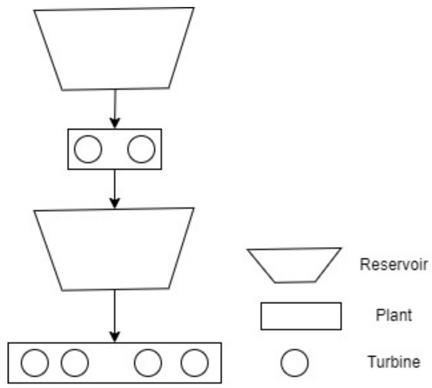

Figure 7: Topology of the hydro system in case B

Table 3 reports the profit obtained in euros in each instance with the MINLP model, the heuristic method, the solution time, and the difference between the two models. The results show that in 10 instances of the stochastic MINLP, no results were obtained with the BONMIN solver because the LP relaxation is infeasible or too expensive. However, the heuristic method provides the result after a few iterations. The solution time is too long to determine that the instance has no solution, so their time was not recorded in the table nor considered in the calculations. The average solution time for the nonlinear model is 131.35 seconds and for the heuristic method with an average of 4 iterations is 86.06 seconds. The MINLP model has an average of 0.08% better results than the heuristic method. Figure 8 is a histogram of the difference percentage between the objective function in the MINLP model and the heuristic method. As mentioned, the MINLP model provides no result in 10 instances. So out of the remaining 50, in 27 instances, their percentage difference is between -0.05% and 0.05%; in 13 instances, the percentage difference is between -0.05% and 0.15%.

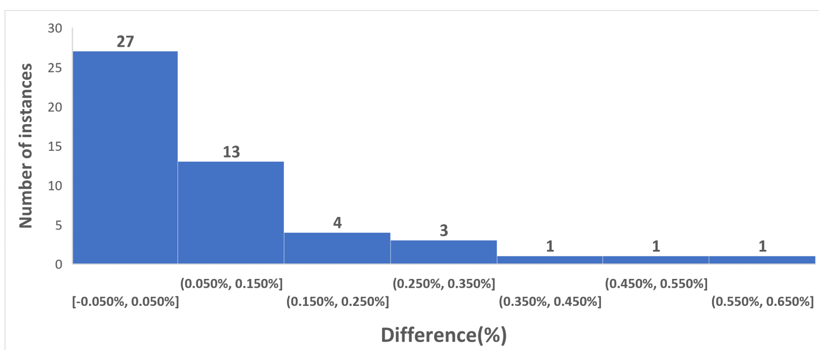

Figure 8: Histogram of difference between stochastic MINLP model and stochastic heuristic method, case B

4.3 Validation results with SHOP

To validate the MINLP and the heuristic method, the value of their objective function has been compared with SHOP. The objective functions and solution methods of these models are not the same; One of the differences is that in the MINLP model and heuristic method, the power produced and the number of active turbines are determined in the first phase, the loading problem. In the second phase, the problem of unit commitment is solved, and the start-up costs are penalized; therefore, the start-up costs are not a decision variable in the bidding problem in the MINLP model and heuristic method, and it is optimized in the second phase. But in SHOP model, the start-up costs are a part of the

Table 3: Results for MINLP and heuristic methods for case B

Instance	Inflows	MINLP (EUR)	Heuristic (EUR)	Difference (%)	Time MINLP (s)	Time HEU (s)	Diff Time (s)	Number of iterations
1	Low	415,640	415,563	0.019%	146.03	72.40	73.63	4
2	Low	NO Reslut	525,654	-	-	54.96	-	3
3	Low	488,798	488,760	0.008%	160.38	74.95	85.43	4
4	Low	439,840	439,847	-0.002%	94.11	188.17	-94.06	5
5	Low	458,945	458,933	0.003%	229.52	220.56	8.96	5
6	Low	519,198	519,282	-0.016%	69.63	80.99	-11.36	4
7	Low	583,800	583,800	0.000%	15.61	31.87	-16.27	3
8	Low	382,630	382,642	-0.003%	173.33	64.34	108.99	4
9	Low	442,410	442,356	0.012%	18.32	66.39	-48.07	4
10	Low	411,106	410,746	0.088%	194.91	78.30	116.61	4
11	Low	432,330	432,409	-0.018%	123.93	87.46	36.47	5
12	Low	NO Reslut	458,164	-	-	68.55	-	4
13	Low	528,858	528,845	0.002%	36.43	50.25	-13.82	3
14	Low	506,363	506,398	-0.007%	62.90	64.37	-1.47	4
15	Low	463,316	463,207	0.024%	84.71	71.04	13.67	4
16	Low	495,952	495,652	0.060%	39.36	67.60	-28.24	4
17	Low	443,889	443,863	0.006%	299.39	98.47	200.93	4
18	Low	NO Reslut	482,266	-	-	117.64	-	4
19	Low	559,790	559,793	-0.001%	98.22	102.68	-4.46	4
20	Low	370,581	370,236	0.093%	149.08	97.89	51.18	5
21	Medium	440,527	440,524	0.001%	180.08	85.63	94.45	5
22	Medium	503,192	503,176	0.003%	27.37	85.83	-58.46	5
23	Medium	325,549	324,584	0.296%	95.65	106.05	-10.40	4
24	Medium	501,095	500,770	0.065%	396.96	110.18	286.78	4
25	Medium	557,098	557,083	0.003%	71.98	51.61	20.38	3
26	Medium	307,438	307,020	0.136%	106.44	126.43	-20.00	5
27	Medium	NO Reslut	372,167	-	-	95.76	-	5
28	Medium	453,719	453,868	-0.033%	204.04	92.29	111.75	5
29	Medium	314,211	314,292	-0.026%	152.83	43.83	108.99	4
30	Medium	429,609	429,519	0.021%	163.47	96.27	67.20	5
31	Medium	NO Reslut	356,977	-	-	107.10	-	6
32	Medium	411,493	410,729	0.186%	499.41	67.92	431.49	4
33	Medium	388,272	387,185	0.280%	104.42	70.31	34.10	4
34	Medium	433,909	434,010	-0.023%	94.48	56.30	38.18	5
35	Medium	571,885	571,872	0.002%	20.73	59.18	-38.45	2
36	Medium	NO Reslut	416,822	-	-	73.38	-	4
37	Medium	482,433	482,012	0.087%	90.46	82.68	7.78	4
38	Medium	418,097	417,490	0.145%	221.52	78.69	142.83	4
39	Medium	374,595	374,287	0.082%	57.20	197.34	-140.14	5
40	Medium	493,862	494,090	-0.046%	221.95	95.48	126.47	5
41	High	544,617	543,946	0.123%	32.99	69.76	-36.78	4
42	High	NO Reslut	337,140	-	-	97.11	-	6
43	High	398,340	397,600	0.186%	97.53	102.38	-4.85	6
44	High	383,772	383,187	0.152%	66.77	139.29	-72.52	4
45	High	NO Reslut	404,323	-	-	66.24	-	4
46	High	488,761	488,767	-0.001%	33.58	51.60	-18.02	3
47	High	NO Reslut	416,091	-	-	86.60	-	5
48	High	484,657	484,528	0.027%	182.25	70.42	111.83	4
49	High	406,413	406,616	-0.050%	50.83	66.80	-15.97	4
50	High	432,191	431,556	0.147%	133.11	77.25	55.86	4
51	High	404,549	402,236	0.572%	126.90	72.00	54.89	4
52	High	371,200	369,391	0.487%	144.67	66.16	78.51	4
53	High	401,446	400,178	0.316%	363.22	86.55	276.67	5
54	High	NO Reslut	377,770	-	-	46.51	-	4
55	High	407,531	407,105	0.105%	106.97	66.96	40.01	4
56	High	390,692	390,313	0.097%	85.52	82.09	3.42	5
57	High	447,669	447,465	0.046%	146.40	108.32	38.08	6
58	High	361,972	361,237	0.203%	92.69	67.71	24.98	4
59	High	322,351	321,070	0.397%	163.17	64.46	98.71	4
60	High	401,820	401,520	0.075%	35.95	87.43	-51.48	5

objective function. Another difference is that the water value is included in the objective function of SHOP, but in the MINLP model and the heuristic method, the final volume is one of the input data. Therefore, for comparison, all conditions, such as the initial and final volume of the reservoir, are the same. We consider zero for the start-up cost and water value in all cases examined. The inflow is considered into three categories: low, medium and high. The comparison results of the MINLP model and heuristic method with SHOP are shown in Table 4.

Table 4: Validation results with SHOP in case B

Instance	Inflow	SHOP (EUR)	MINLP (EUR)	Improvement %	Heuristic (EUR)	Improvement %
1	Low	490,117	-	-	491,371	0.26%
2	low	288,609	-	-	291,975	1.17%
3	low	388,676	396,721	2.07%	396,603.0	2.04%
4	Medium	392,105	-	-	402,791	2.73%
5	Medium	518,067	-	-	518,067	0.00%
6	Medium	468,489	-	-	474,081	1.19%
7	High	513,283	513,283	0.00%	513,283	0.00%
8	High	477,594	485,393	1.63%	485,393	1.63%
9	High	320,898	-	-	324,437	1.10%

The improvement percentage of the objective function value in the MINLP model and the heuristic method compared to SHOP is obtained from Equations (41) and (42), and it is displayed in Table 4 in the side column of the objective function of each model. As noted, because the formulation of the models is different from SHOP, the improvement of the objective function may not provide a totally right picture of the benefits, and it is only used to validate the results. Figure 9 shows the comparison of the bid curves for instance 3 in the MINLP model and the heuristic method with SHOP at 5, 9, 19, and 21 hours. Due to the different models and their solvers, the results are different in some hours and very similar in others.

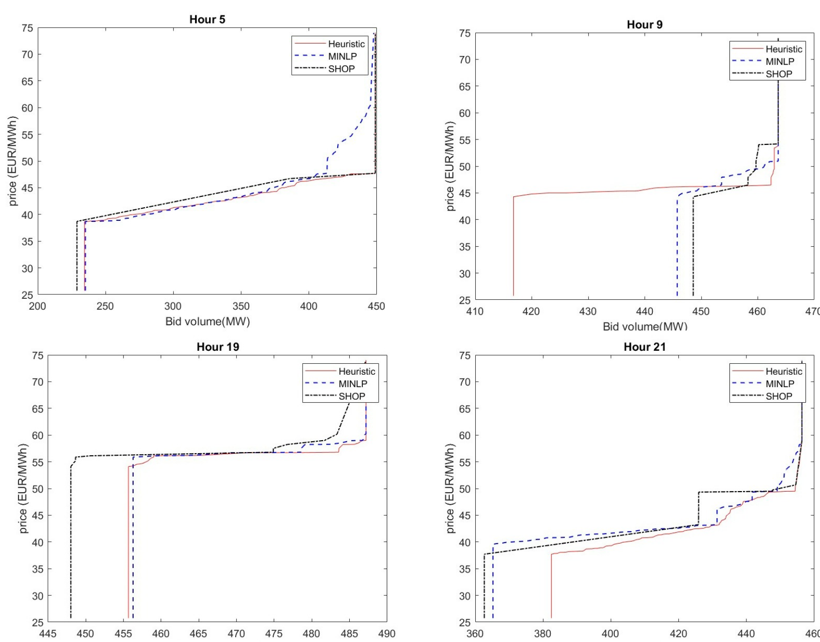

Figure 9: Bid curves for a selected hour (Hours 5, 9, 19, 21)

5 Conclusion

This paper introduced a two-stage mixed integer nonlinear stochastic model to obtain the optimal bids for a price-taking hydropower producer in the day-ahead market. Since hydropower optimization is generally a nonlinear problem, a MINLP model was used to determine the optimal power production in the bidding problem. Instead of using the combination of turbines, the maximum power output surface was used to reduce the complexity of the problem. The results of the case studies showed that MINLP model was efficient for small problems and could be solved in a short time. However, for larger problems, the solution time increased, and the binary variables and the complexity of the nonlinear problem make the MINLP model unable to find the solution in some instances. Therefore, a heuristic method was proposed to solve the two-stage mixed integer nonlinear stochastic model, which can reach the appropriate solution after several iterations in a short time. Average revenues in case studies A and B are 0.014% and 0.08% higher, respectively, in the MINLP model than in the heuristic method. The presented methods were tested according to the available data. For future studies, another data set with more powerhouses in a larger system will be investigated, and uncertainty of inflows and prices will be considered simultaneously in the model. The heuristic method can be combined with metaheuristic methods, and the results will be compared.

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