Les Cahiers du GERAD ISSN: 0711–2440

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G–2023–58

November 2023

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2023
– Bibliothèque et Archives Canada, 2023

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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– Library and Archives Canada, 2023

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Dynamic programming for valuing options embedded in corporate bonds

Malek Ben-Abdellatif [a]

Hatem Ben-Ameur [b]

Rim Chérif [c, d]

Tarek Fakhfakh [e]

[a] *ESLSCA University, Giza Governorate, Egypt*

[b] *Department of Decision Sciences, HEC Montréal & GERAD, Montréal, (Qc), Canada, H3T 2A7*

[c] *Department of Decision Sciences, HEC Montréal, Montréal, (Qc), Canada, H3T 2A7* à

[d] *School of Business, American University in Cairo, New Cairo, Egypt*

[e] *Finance Department, FSEG Sfax, Sfax 3018, Tunisia*

hatem.ben-ameur@hec.ca

November 2023
Les Cahiers du GERAD
G–2023–58

Abstract : We consider a structural model to design and evaluate the American call, conversion, and put options embedded in corporate bonds. We use dynamic programming and finite elements to efficiently solve the setting. We show that the option exercise policies can be characterized via a set of exercise thresholds. We achieve a sensitivity analysis of the option values with respect to the model parameters, and document on the default barriers and the exercise thresholds.

Keywords : Structural models, corporate securities, options embedded in corporate bonds, dynamic programming, finite elements

Acknowledgements: This paper was supported by two research grants received by the second author from the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Canadian Statistical Sciences Institute (CANSSI).

1 Introduction

We use dynamic programming (DP) and finite elements to design and evaluate the American call, conversion, and put options embedded in corporate bonds. The call and put prices as well as the conversion ratios and factors are key design parameters. We show that redemption, retraction, and conversion can be characterized by a set of exercise thresholds. We approximate the value functions of corporate securities, and analyse their sensitivities with respect to the model parameters. We also document on the firm default barriers and the option exercise thresholds.

DP combined with finite elements in our context turns out to be highly efficient for multiple reasons. Part of the experiment is achieved in closed form via a set of transition parameters that are available under alternative Lévy processes. See Ayadi et al. (2016) and Ben-Ameur et al. (2023) for a derivation under the lognormal process, jump-diffusion processes, and the variance-gamma process. The use of the (true) transition parameters along with local interpolations results in efficient approximations of the value functions of corporate securities at each step of the backward recursion. DP assumes only a space, but not a time discretization. This is advantageous in case of long protection periods and distant exercise opportunities. Moreover, DP supports only a numerical, but not a statistical error.

The call option gives the firm the right to call back the host bond at or before its maturity at a given call price, while the put option (the conversion option) provides the investor with the right to return the host bond to its issuer (to convert the bond into shares of the issuer's equity) at or before its maturity at a given put price (a given conversion factor). Options embedded in corporate bonds are usually of American style. See Wilson and Fabozzi (1996) for further details.

According to the Securities Industry and Financial Markets Association (SIFMA), the principal amount of US corporate bonds issued in 2021 is about $1,962 billion, 87% of which host one or multiple embedded options. This proportion has remained relatively high (above 75% over the last five years). According to the Mergent Fixed Income Securities Database (FISD), US corporate bonds are mostly senior, but their convertible versions are mainly subordinated and host in addition the embedded call option. This is also documented in Krishnaswami and Yaman (2008).

Except for a few cases for which the explicit approach is successful, numerical procedures have been used to evaluate options embedded in bonds, including the lattice approach, finite differences, and dynamic programming. Ingersoll (1977) develops a closed-form solution for a callable and convertible pure bond under Merton's (1974) assumptions. Along the same lines, Lewis (1991) and Bühler and Koziol (2002) extend Ingersoll's (1977) findings to several seniority classes and two conversion dates, respectively, while Koziol (2006) considers a convertible console debt under Leland's (1994) assumptions. Zhu et al. (2018) considers a one-dimensional model for an underlying stock along with a riskless pure bond with its embedded put and conversion options. The authors develop an integral representation of the host-bond value, which is solved by numerical integration. Lin and Zhu (2022) extend the setting of Zhu et al. (2018) to embedded call options.

The lattice approach has been widely used to evaluate American options embedded in corporate bonds. Derman (1994) proposes a (semi) rational binomial tree for an underlying stock, where the default event is implicitly accounted for via a shift of the risk-free discount rate consistent with the observed credit spread of the host bond. Along the same lines, Ammann et al. (2003) and Ma et al. (2020) build on Derman's (1994) idea and suggest a one- and two-dimensional binomial trees, respectively. Carayanopoulos and Kalimipalli (2003) propose a one-dimensional trinomial tree consistent with the reduced-form construction of Duffie and Singleton (1999). Similarly, Hung and Wang (2002) consider a reduced-form two-dimensional binomial tree à la Jarrow and Turnbull (1995) with non-correlated state variables, while Chambers and Lu (2007) consider an extension with correlated state variables. The lattice approach has also been used to design two-dimensional structural models and evaluate risky host bonds with their embedded options (Acharya and Carpenter 2002, Das and Sundaram 2007, ElKamhi et al. 2012, and Dai et al. 2022).

Brennan and Schwartz (1977 and 1980) are pioneers at using (backward) finite differences to evaluate American options embedded in risky bonds in one- then two-dimensional structural models, where the first (second) state variable is the firm asset value (risk-free rate) that moves according to a log-normal (a Gaussian mean-reverting process). Likewise, Carayanopoulos (1996) solves a version of Brennan and Schwartz's (1980) setting for which the risk-free rate moves according to a root-square mean-reverting process. The same approach is used to solve one-dimensional reduced-form models, and evaluate LYON pure bonds (McConnell and Schwartz 1986) and coupon bonds (Takahashi et al. 2001) with their embedded options, while Tsiveriotis and Fernandes (1998) build on Derman's (1994) idea to account for default. Multiple refinements of finite differences have been proposed in one- and two-dimensional continuous settings (Kim et al. 1993, Barone-Adesi et al. 2003, Bermúdez and Nogueiras 2004, Yang et al. 2018, and Lin and Zhu 2020).

Ballotta and Kyriakou (2015) are pioneers at using (backward) dynamic programming to evaluate American options embedded in risky bonds. They proposed a two-dimensional structural model, where the firm asset value moves according to a jump-diffusion process and the risk-free rate as a mean-reverting diffusion process. Fast Fourrier transforms are used to approximate the value functions of corporate securities at each step of the backward recursion, including the host bond with its embedded options.

Ammann et al. (2008) design a Monte Carlo experiment to evaluate American options embedded in risky bonds in a one-dimensional model, where the sole state variable is an underlying stock. The default event is accounted for as in Tsiveriotis and Fernandes (1998) and Derman (1994). Their numerical procedure consists of an iterative search of the options exercise regions, where the host-bond value is estimated at each step then optimized. It can be classified as a forward DP approach. A numerical/empirical investigation is conducted under a GARCH setting.

A new research stream based on financial modeling and artificial intelligence has recently arisen, where traditional constructions have been reworked to integrate machine learning-based techniques. Tan et al. (2022) combine the empirical and the rational approach, and design a model in the spirit of Tsiveriotis and Fernandes (1998) and Derman (1994). Then, they train a neural network to predict the values of embedded options.

No-arbitrage pricing in structural models can be seen as an optimal Markov process since the value functions of corporate securities are known at a given future date, here the firm debt maturity, in addition to being forward looking. The value function of corporate securities depend on the future evolution of the market and its current position, but not on its past history. Thus, dynamic programming (DP) can be used to solve the model backward in time starting from the debt maturity down to the origin. DP splits the evaluation problem into time steps $[t_n, t_{n+1}]$, for $n = N-1, \ldots, 0$, over each one, it alternates between value-function interpolation at t_{n+1} and value-function integration at t_n to solve the model backward in time. At step n, DP assumes that all corporate securities have already been evaluated and all decisions have already been made optimally from t_{n+1} until maturity $t_N = T$, for all levels of the state process, and that the firm has not been liquidated yet and its embedded options have not been activated yet. Thus, if all decisions are made optimally at node (t_n, a), where t_n is the current evaluation/decision date and $a = A_{t_n}$ is the level of the state variable at t_n, then all decisions must have been taken optimally from that node until maturity $t_N = T$, independently of the firm past trajectory. DP is combined with finite elements at each step of the backward recursion to efficiently approximate the value functions of corporate securities. Valuing American options embedded in corporate bonds is done via a difference analysis between paired scenarios on the host bond with and without its embedded options.

The rest of this paper is organized as follows. Section 2 presents the model design and resolution. Section 3 discusses the options embedded in corporate bonds and their (early) exercise decisions. Section 4 exhibits a numerical investigation and Section 5 concludes.

2 The model

We consider a public company with an option-free senior debt and a host bond with its American-style embedded call, conversion, and put options. The firm balance-sheet equality at (t, a) is

$$a + \mathrm{TB}_t(a) - \mathrm{BC}_t(a) = D_t^s(a) + D_t^h(a) + \mathcal{E}_t(a), \quad (1)$$

where $a = A_t$ is the level of the state process A at time $t \in [0, T]$, while T is the maturity of the overall debt portfolio. The corporate securities A, TB, BC, D^s, D^h, $D = D^s + D^h$, and $\mathcal{E}$ represent the value functions of the firm's assets, tax benefits, bankruptcy costs, senior bond potfolio, host bond, overall debt portfolio, and equity, respectively. Under survival, the firm pays coupons and save taxes at the corporate tax rate r^c, while, under default, it supports proportional bankruptcy costs at the rate $w \in [0, 1]$. The firm is committed to paying $d_n = d_n^s + d_n^h$ at the payment date t_n to its creditors, where d_n^s and d_n^h are the outflows generated by the senior debt and the host bond, respectively. The set of payment dates is $\mathcal{P} = \{t_1, \ldots, t_N = T\}$. The total outflow d_n includes interest payments $C_n = C_n^s + C_n^h$ as well as capital payments $P_n = P_n^s + P_n^h$. The quantities C_n^s, C_n^h, P_n^s, and P_n^h are known to all investors from the very beginning. No payment takes place at the present date $t_0 = 0$. Assume that the host bond is junior and has a longer maturity than the senior bond. The case of a senior host bond can be handled with a minor modification. We also assume that (early) exercise decisions are taken only at coupon/principal payment dates.

Table 1 and Table 2 explicit the value functions of corporate securities without embedded options at and before the debt maturity as in Ayadi et al. (2016) and Ben-Ameur et al. (2023).

Table 1: DP value functions at maturity without embedded options

BSE	Liquidation $a \le d_N - \mathrm{tb}_N$	Survival $a > d_N - \mathrm{tb}_N$
$+a = A_{t_N}$	a	a
$+\mathrm{TB}(t_N, a)$	0	tb_N
$-\mathrm{BC}(t_N, a)$	$-wa$	0
$=$	$=$	$=$
$+D^s(t_N, a)$	$\min((1-w)a, d_N^s)$	d_N^s
$+D^h(t_N, a)$	$\max(0, (1-w)a - d_N^s)$	d_N^h
$+\mathcal{E}(t_N, a)$	0	$a - (d_N - \mathrm{tb}_N)$

No-arbitrage results in

$$\begin{aligned} & \mathbb{E}_{na}^*\left[\rho_n A_{t_{n+1}}\right] + \mathbb{E}_{na}^*\left[\rho_n \mathrm{TB}_{t_{n+1}}\left(A_{t_{n+1}}\right)\right] - \mathbb{E}_{na}^*\left[\rho_n \mathrm{BC}_{t_{n+1}}\left(A_{t_{n+1}}\right)\right] \\ = \; & \mathbb{E}_{na}^*\left[\rho_n D_{t_{n+1}}^s\left(A_{t_{n+1}}\right)\right] + \mathbb{E}_{na}^*\left[\rho_n D_{t_{n+1}}^h\left(A_{t_{n+1}}\right)\right] + \mathbb{E}_{na}^*\left[\rho_n \mathcal{E}_{t_{n+1}}\left(A_{t_{n+1}}\right)\right], \end{aligned}$$

where $\rho_n = e^{-r(t_{n+1}-t_n)}$ is the risk-free discount factor over $[t_n, t_{n+1}]$ and $\mathbb{E}_{na}^*[.] = \mathbb{E}^*[. \mid A_{t_n} = a]$ is the conditional expectation operator under a risk-neutral probability measure given $A_{t_n} = a$. The balance-sheet equality under survival and continuation at t_n is

$$\begin{aligned} a + \left[\overline{\mathrm{TB}}_{t_n}(a) + \mathrm{tb}_n\right] - \left[\overline{\mathrm{BC}}_{t_n}(a)\right] = & \left[\overline{D}_{t_n}^s(a) + d_n^s\right] + \left[\overline{D}_{t_n}^h(a) + d_n^h\right] \\ & + \left[\overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right], \end{aligned}$$

where $\overline{v}_{t_n}(a) = \mathbb{E}_{na}^*\left[\rho_n v_{t_{n+1}}\left(A_{t_{n+1}}\right)\right]$ represents the value function of a corporate security at (t_n, a) based on its future potentialities and $v_{t_n}(a)$ its overall value function, including its current cash flows. Thus, the balance-sheet equality under survival at (t_n, a) results from the one at t_{n+1} and no-arbitrage pricing. The positive amount $d_n - \mathrm{tb}_n$ is the due debt payment d_n net of the tax benefits $\mathrm{tb}_n = r^c \times C_n$ under survival at t_n. The survival condition at (t_n, a) comes

$$\mathcal{E}_{t_n}(a) = \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n) > 0, \quad (2)$$

which clearly shows that $\overline{\mathcal{E}}_{t_n}(a) > \mathcal{E}_{t_n}(a)$. To finance the due debt payment $d_n = P_n + C_n$ net of the tax benefits tb_n at (t_n, a), the firm issues new equity shares, while the price per share remains constant. Since $\overline{\mathcal{E}}_{t_n}(a)$ is a continuous and increasing function of $a = A_{t_n}$, the survival region can be expressed as $\{a > b_n\}$, where b_n is the endogenous default barrier at t_n (to be determined). Table 2 reports the option-free value functions of corporate securities before maturity.

Table 2: DP value functions before maturity without embedded options

BSE	Liquidation $a \leq b_n$	Survival $a > b_n$
$+a = A_{t_n}$	a	a
$+\text{TB}(t_n, a)$	0	$\overline{\text{TB}}(t_n, a) + \text{tb}_n$
$-\text{BC}(t_n, a)$	$-wa$	$-\overline{\text{BC}}(t_n, a)$
$=$	$=$	$=$
$+D^s(t_n, a)$	$\min\left[(1-w)a, \overline{D}^s(t_n, a) + d_n^s\right]$	$\overline{D}^s(t_n, a) + d_n^s$
$+D^h(t_n, a)$	$\max\left[0, (1-w)a - D^s(t_n, a)\right]$	$\overline{D}^h(t_n, a) + d_n^h$
$+\mathcal{E}(t_n, a)$	0	$\overline{\mathcal{E}}(t_n, a) - (d_n - \text{tb}_n)$

The survival condition $\overline{\mathcal{E}}_{t_n}(a) > (d_n - \text{tb}_n)$ is equivalent to saying that $a = A_{t_n} > b_n$, where b_n is the endogenous default barrier at t_n. This holds true in Lévy settings. For consistency, we rediscuss the balance-sheet equality under survival at (t_N, a). One has

$$\begin{aligned}A_{t_N} + \left[\overline{\text{TB}}_{t_N}(a) + \text{tb}_N\right] - \left[\overline{\text{BC}}_{t_N}(a)\right] = \left[\overline{D}^s_{t_N}(a) + d^s_N\right] + \left[\overline{D}^j_{t_N}(a) + d^h_N\right] \\ + \left[\overline{\mathcal{E}}_{t_N}(a) - (d_N - \text{tb}_N)\right],\end{aligned}$$

where $\overline{\text{TB}}_{t_N}(a) = 0$, $\overline{\text{BC}}_{t_N}(a) = 0$, $\overline{D}^s_{t_N}(a) = 0$, $\overline{D}^h_{t_N}(a) = 0$, and $\overline{\mathcal{E}}_{t_N}(a) = a$ given the assumption of an all-equity firm under survival from maturity on. The survival condition at maturity comes $\{a = A_{t_N} > b_N = d_N - \text{tb}_N\}$, which is in line with Table 1.

Assume now that the model has been solved backward in time from $t_N = T$ to t_{n+1}, should we set $t_{n+1} = t_N$, which results in an approximation $\widetilde{v}_{t_{n+1}}$ of $\overline{v}_{t_{n+1}}$ on a mesh of grid points $\mathcal{G} = \{a_1, \ldots, a_p\}$ that span the overall state space $\mathbb{R}^*_+$, with the convention that $a_0 = 0$ and $a_{p+1} = +\infty$. Set $\widetilde{v}_{t_N} = v_{t_N}$ on $\mathcal{G}$ at maturity. To extend $\widetilde{v}_{t_{n+1}}$ from $\mathcal{G}$ to $\mathbb{R}^*_+$, we use a piecewise linear interpolation $\widehat{v}_{t_n}$, defined as follows:

$$\widehat{v}_{t_{n+1}}(a) = \sum_{i=0}^{p} \left(\alpha_i^{n+1} + \beta_i^{n+1} a\right) \times \mathbb{I}\left(a_i \leq a < a_{i+1}\right),$$

where the α_i^{n+1}'s and the β_i^{n+1}'s are its local coefficients and $\mathbb{I}()$ is the indicator function. The local coefficients are obtained by solving the sytem of linear equations $\widehat{v}_{t_{n+1}} = \widetilde{v}_{t_{n+1}}$ on each subinterval. No-arbitrage pricing allows one to move backward in time and approximate $\overline{v}_{t_n}(a_k)$ on $\mathcal{G}$ by

$$\begin{aligned}\widetilde{v}_{t_n}(a_k) =& \mathbb{E}^*_{na_k}\left[\rho_n \widehat{v}_{t_{n+1}}\left(A_{t_{n+1}}\right)\right] \\ =& \rho_n \sum_{i=0}^{p} \alpha_i^{n+1} \times T^0_{k,i} + \beta_i^{n+1} \times T^1_{k,i},\end{aligned} \tag{3}$$

where $T^{\nu}_{k,i} = \mathbb{E}^*_{na_k}\left[A^{\nu}_{t_{n+1}} \times \mathbb{I}\left(a_i \leq A_{t_{n+1}} < a_{i+1}\right)\right]$, for $\nu \in \{0, 1\}$, are transition parameters of the underlying Markov process. For example, $T^0_{k,i}$ is the transition probability that the underlying process moves from a_k at t_n and visits the interval $[a_i, a_{i+1}]$ at t_{n+1}. These parameters are known in closed for several Lévy processes. They remain constant for homogenous Markov dynamics as long as $\Delta t_n = t_{n+1} - t_n$ and $\mathcal{G}$ are fixed. From the perspective of an investor at (t_n, a_k), formula (3) decomposes $\overline{v}_{t_n}(a_k)$ into small future value pieces (the α_i^{n+1}'s and β_i^{n+1}'s) multiplied by their associated transition parameters (the $T^0_{k,i}$'s and $T^1_{k,i}$'s), sums the products, and discounts it back from t_{n+1} to t_n. Table 1,

Table 2, and formula (3) result in the DP value functions $\widetilde{v}_{t_n}$, defined on $\mathcal{G}$, and $\widehat{v}_{t_n}$, defined on $\mathbb{R}_+^*$, along with their Greeks Δ, Γ, and Θ. A second run is required to approximate v and ρ. The rest comes by backward induction.

3 The embedded options

We herein discuss each option exercise policy alone, then we specify their common impact on the firm balance sheet. We assume that conversion and retraction have priority on redemption, but other conventions can be analysed as well. A value function in the form $\overline{v}_{t_n}(a) = \mathbb{E}_{na}^*\left[\rho_n v_{t_{n+1}}\left(A_{t_{n+1}}\right)\right]$ represents now the future potentialities of a corporate security in the presence of one or multiple options embedded in the host bond. One can make the parallel with the holding value of an American vanilla option. The following notation is case sensitive.

3.1 The call option (redemption)

A callable bond provides its issuer with the privilege of redeeming the host bond from its holder at or before maturity for a known call price. The host bond is redeemed at (t_n, a) if its value, based on its future and present potentialities, exceeds its exercise value on redemption:

$$\overline{D}_{t_n}^h(a) + P_n^h + C_n^h \geq c_n + C_n^h, \quad \text{for } t_n \in \mathcal{P}, \tag{4}$$

where c_n is the call price at t_n. It is worth noticing that the call price c_n is compared to the bond value net of the current interest reimbursement C_n^h at t_n. Given that $\overline{D}_{t_n}^h(a)$ is a continuous and increasing function of $a = A_{t_n}$ in Lévy models, redemption takes place when $a = A_{t_n}$ is higher than a certain call threshold. See Figure 3 for an illustration.

As redemption has no impact on the value of the portfolio made of the host bond and equity, one has

$$D_{t_n}^h(a) = c_n + C_n^h$$

and

$$\mathcal{E}_{t_n}(a) = \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n) + \left(\overline{D}_{t_n}^h(a) + P_n^h - c_n\right),$$

on redemption, which reinforces the firm healthiness. New equity shares quoted at the same price are issued to the benefit of equity holders. The same expressions hold at maturity, with the convention that $\overline{\mathcal{E}}_{t_N}(a) = a$ and $\overline{D}_{t_N}^h(a) = 0$ to say that the firm becomes an all-equity firm from debt maturity on. The call price c_N is usually set at the host-bond principal amount P_N^h so that redemption is neutralized at debt maturity.

3.2 The conversion option

A convertible bond provides its holder with the privilege of converting the host bond at or before maturity into equity shares according to a known convertion ratio or factor. The host bond is converted into equity shares at (t_n, a) when its exercise value based on conversion exeeds its value based on its future and present potentialities, that is,

$$\overline{D}_{t_n}^h(a) + P_n^h + C_n^h \leq m_{t_n}(a) \times u_{t_n}(a) + C_n^h,$$

where $u_{t_n}(a)$ is the price per share on conversion and $m_{t_n}(a)$ is the number of shares issued on conversion, namely, the conversion ratio.

As conversion has no impact on the value of the portfolio made of the firm's host bond and equity, one has

$$\overline{D}_{t_n}^h(a) + P_n^h + C_n^h + M_{t_n}(a) \times U_{t_n}(a) = m_{t_n}(a) \times u_{t_n}(a) + C_n^h + M_{t_n}(a) \times u_{t_n}(a),$$

where $M_{t_n}(a)$ is the number of shares, $U_{t_n}(a)$ the price per share, and $M_{t_n}(a) \times U_{t_n}(a) = \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n)$ the value of equity on holding. The last equation and the conversion condition highlight the dilution effect on conversion, that is,

$$u_{t_n}(a) \leq U_{t_n}(a).$$

The fact that

$$u_{t_n}(a) = \frac{1}{M_{t_n}(a) + m_{t_n}(a)} \left[\overline{D}_{t_n}^h(a) + P_n^h + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n) \right]$$

provides the conversion condition

$$\overline{D}_{t_n}^h(a) + P_n^h \leq \kappa_n \left(\overline{D}_{t_n}^h(a) + P_n^h + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n) \right), \tag{5}$$

which results in the following value functions of the host bond and equity

$$D_{t_n}^h(a) = \kappa_n \left(\overline{D}_{t_n}^h(a) + P_n^h + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n) \right) + C_n^h$$

and

$$\mathcal{E}_{t_n}(a) = (1 - \kappa_n) \left(\overline{D}_{t_n}^h(a) + P_n^h + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n) \right)$$

where

$$\kappa_n(a) = \frac{m_{t_n}(a)}{M_{t_n}(a) + m_{t_n}(a)} \in [0, 1],$$

is the fraction of shares obtained by the host-bond holders on conversion, namely, the conversion factor. The conversion factors are usually assumed to be fixed and known in advance. They are simply indicated by κ_n or κ. The value function on conversion of the portfolio made of the host bond (net of its current coupon) and equity, that is, $\overline{D}_{t_n}^h(a) + P_n^h + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \text{tb}_n)$, is shared between the new equity holders (old host-bond holders) and old equity holders. Clearly, conversion strengthens the firm healthiness. Again, the above expressions hold at maturity with the property that $\overline{\mathcal{E}}_{t_N}(a) = a$ and $\overline{D}_{t_N}^h(a) = 0$.

Given Equation (5), conversion is activated more frequently for high κ than for low κ (ceteris paribus). This is perfectly consistent with Ingersoll's (1977) conversion condition

$$a \geq \frac{P_N}{\kappa},$$

when the overall debt portfolio is reduced to a convertible pure bond, in which case, one has $P_N^h = P_N$, $C_N^h = C_N = 0$, $a = A_{t_N}$, $\text{TB}_{t_N}(a) = 0$, $\text{BC}_{t_N}(a) = 0$, $\overline{D}_{t_N}^h(a) = \overline{D}_{t_N}(a) = 0$, and $\overline{\mathcal{E}}_{t_N}(a) = a$.

There is no obvious theoretical argument that the conversion region can be characterized by a conversion threshold. Although our numerical investigation shows that conversion is optimally activated when the asset value is higher than a certain threshold, we keep checking for optimal conversion from the default barrier on. See Figure 3 for an illustration.

3.3 The put option (retraction)

A putable bond provides its holder with the privilege of retracting (retrurning) the host bond to its issuer at or before maturity for a known put price. The host bond is returned at (t_n, a) when its value based on retraction exceeds its value based on its future and present potentialities, that is,

$$\overline{D}_{t_n}^h(a) + P_n^h + C_n^h \leq p_n + C_n^h, \tag{6}$$

where p_n is the put price at t_n. New equity shares are issued with a dilution effect to finance the benefit to the host-bond holders on retraction. The value functions of the host bond and equity on retraction become

$$D^h_{t_n}(a) = p_n + C^h_n,$$

and

$$\mathcal{E}_{t_n}(a) = \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n) - \left[p_n - \left(\overline{D}^h_{t_n}(a) + P^h_n\right)\right],$$

which tends to increase the default probability. The same expressions hold at maturity with the convention that $\overline{\mathcal{E}}_{t_N}(a) = a$ and $\overline{D}^h_{t_N}(a) = 0$. The put price p_N is usually set at the principal amount P^h_N so that retraction is neutralized at the debt maturity.

The put option has been used as protection against takeovers, but it can drastically weaken the firm healthiness upon retraction. David (2001) discusses the strategic value of poison puts in a game-theory framework. The author reports that junior host-bond holders can force concessions from equity holders under survival, and appropriate the most liquid assets under liquidation. A parallel can be made in our setting if we account for reorganization processes and illiquidity costs as new intangible assets in the firm's balance sheet. This is left for a future research investigation. For simplicity, we assume that the put option cannot provoke default.

Given that $\overline{D}^h_{t_n}(a)$ is a continuous and an increasing function of $a = A_{t_n}$ under Lévy processes, the retraction region at t_n is bounded above by a certain put threshold. See Figure 3 for an illustration.

3.4 The exercise decisions

We assume that the host-bond holders have priority over equity holders on exercise decisions. We also assume that $p_n \leq c_n$, for $t_n \in \mathcal{P}$, and that $p_N = c_N = P^h_N$. In case the host bond is protected against exercise decision(s) at certain payment dates $t_n \in \mathcal{P}$, set $p_n = \kappa_n = 0$ and/or $c_n = \infty$. The fact that $p_n \leq c_n$, for $t_n \in \mathcal{P}$, results in distinct redemption and retraction regions, while both exercise regions can intersect with conversion regions. See Figure 2 in Section 4 for a numerical illustration.

Case 1. Redemption is profitable for equityholders when

$$\overline{D}^h_{t_n}(a) + P^h_n \geq c_n,$$

which results in

1. an optimal conversion whenever conversion is profitable for the host-bond holders:

$$c_n \leq \overline{D}^h_{t_n}(a) + P^h_n \leq \kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right];$$

2. a forced conversion when conversion is not profitable for host-bond holders, but represents a less bad outcome than redemption:

$$c_n \leq \kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right] \leq \overline{D}^h_{t_n}(a) + P^h_n;$$

3. an optimal redemption when convertion further deteriorates the position of the host-bond holders:

$$\kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right] \leq c_n \leq \overline{D}^h_{t_n}(a) + P^h_n.$$

Case 2. Retraction is profitable for the host-bond holders when

$$D^h_{t_n}(a) + P^h_n \leq p_n,$$

which results in

1. an optimal conversion when conversion is more profitable than retraction for the host-bond holders:
$$\overline{D}^h_{t_n}(a) + P^h_n \le p_n \le \kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right];$$

2. an optimal retraction when retraction is more profitable than conversion for the host-bond holders:
$$\max\left(\overline{D}^h_{t_n}(a) + P^h_n, \kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right]\right) \le p_n,$$
under survival
$$\overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n) - \left[p_n - \left(\overline{D}^h_{t_n}(a) + P^h_n\right)\right] > 0;$$

3. or a continuation (no-exercise).

Case 3. The event
$$p_n \le \overline{D}^h_{t_n}(a) + P^h_n \le c_n$$
results in

1. an optimal conversion whenever conversion is profitable for the host-bond holders:
$$\overline{D}^h_{t_n}(a) + P^h_n \le \kappa_n \left[\overline{D}^h_{t_n}(a) + P^h_n + \overline{\mathcal{E}}_{t_n}(a) - (d_n - \mathrm{tb}_n)\right];$$

2. or a continuation (no-exercise).

4 Numerical investigation

The code lines are written in C, compiled under GCC, and make use of the scientific library GSL to perform specif computing tasks. The experiments are run with a laptop computer equiped with an 8 GB of RAM and an i5 processing CORE.

4.1 DP vs Ingersoll (1977)

Ingersoll (1977) considers a strural setting à la Merton (1974) with a debt made of a pure bond that is callable and convertible at the debt maturity. DP values of convertible bonds, obtained with a coarse mesh of grid points, exactly coincide with their targets, as shown in Table 3. Set $A_0 = \$100$, $N = 1$, $P_1 = P = \$100$, $c_1 = c = \$100$, $p_1 = p = \$0$, and $\kappa_1 = \kappa \in \{0.4, 0.5\}$. The time step $\Delta t = t_1 - t_0 = T = 1$ (year) needs not be small for our DP procedure to run backward and solve the model. Set $\sigma \in \{0.1, 0.2, 0.3\}$ (per year) and $r^f = 0.05$ (per year). Tax benefits and bankruptcy costs are not accounted for ($r^c = 0$ and $w = 0$). PVEO stands for the present value of the embedded option(s).

Table 3: Exact evaluation of convertible pure bonds

	Grid size		$\kappa = 0.4$			$\kappa = 0.5$	
	p	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mathcal{E}_0$	DP–4	24.9136	26.1680	28.8024	24.9136	26.1146	28.2740
	Ingersoll	24.9136	26.1680	28.8024	24.9136	26.1146	28.2740
D_0	DP–4	95.0864	93.8320	91.1976	95.0864	93.8854	91.7260
	Ingersoll	95.0864	93.8320	91.1976	95.0864	93.8854	91.7260
	Merton	95.0864	93.8310	91.1196	95.0864	93.8310	91.1196
	PVEO	0.0000	0.0010	0.0780	0.0000	0.0544	0.6064

This is not surprising since (DP) piecewise linear interpolations can be designed to perfectly recover the value function of each convertible pure bond at maturity, that is, $\widehat{D}_T = D_T$, which makes formula (3) equivalent to Ingersoll's (1977) closed-form solution, that is, $\widehat{D}_0 = D_0$. Figure 1 is an illustration with $\sigma = 0.2$ (per year), $\kappa = 0.5$, and $\mathcal{G} = \{50, 100, 200, 250\}$, where 100 and $200 = 100/0.5$ are the survival barrier and conversion threshold at maturity, respectively.

The PVEO is obtained as a difference between the value of the host bond and its option-free counterpart at (t_0, A_0), while DP solves the model for all $t \in \mathcal{P}$ and $a = A_t > 0$. The higher is κ then the higher is the PVEO. This is consistent with Ingersoll's (1977) exercise (conversion) region $\{a > P/\kappa\}$. Likewise, the higher is σ then the higher is the PVEO. A higher dispersion of the firm.s asset value at maturity increases its probability to exceed the conversion threshold P/κ.

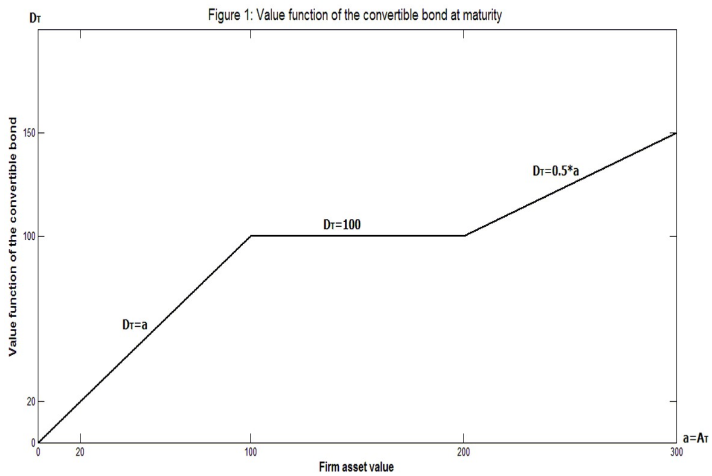

Figure 1

We now experiment under the same set of parameters, except for $N = 52$ and $p \in \{500, 1000, 2000, 4000\}$, while the sole conversion opportunity remains at maturity. Table 4 shows clear convergence of DP approximations to their Ingersoll's (1977) counterparts when the grid size p increases. CPU times for $p = 500$, 1000, 2000, and 4000 are 3, 9, 37, and 151 seconds, respectively.

Table 4: DP vs Ingersoll's (1977) values of convertible bonds

	Grid size p	$\kappa = 0.4$ $\sigma = 0.1$	$\kappa = 0.4$ $\sigma = 0.2$	$\kappa = 0.4$ $\sigma = 0.3$	$\kappa = 0.5$ $\sigma = 0.1$	$\kappa = 0.5$ $\sigma = 0.2$	$\kappa = 0.5$ $\sigma = 0.3$
$\mathcal{E}_0$	DP−500	24.9164	26.1718	28.8067	24.9164	26.1206	28.2809
	DP−1000	24.9159	26.1706	28.8045	24.9159	26.1187	28.2796
	DP−2000	24.9145	26.1694	28.8037	24.9145	26.1157	28.2757
	DP−4000	24.9139	26.1682	28.8027	24.9139	26.1149	28.2744
	Ingersoll	24.9136	26.1680	28.8024	24.9136	26.1146	28.2740
D_0	DP−500	95.0836	93.8282	91.1933	95.0836	93.8794	91.7191
	DP−1000	95.0841	93.8294	91.1955	95.0841	93.8813	91.7204
	DP−2000	95.0855	93.8306	91.1963	95.0855	93.8843	91.7243
	DP−4000	95.0861	93.8318	91.1973	95.0861	93.8851	91.7256
	Ingersoll	95.0864	93.8320	91.1976	95.0864	93.8854	91.7260
	Merton	95.0864	93.8310	91.1196	95.0864	93.8310	91.1196
	PVEO	0.0000	0.0010	0.0780	0.0000	0.0544	0.6064

4.2 DP vs Brennan and Schwartz (1977)

Brennan and Schwartz (1977) consider a structural model à la Geske (1977) with a debt made of a coupon bond that hosts the call and conversion options. Set $N = 40$, $T = t_{40} = 20$ (years), $\Delta t_n =$

$t_{n+1} - t_n = 0.5$ (years), for $n = 0, \ldots, 39$, $P_1 = \cdots = P_{39} = 0$ and $P_{40} = 40$ \$, $C_1 = \cdots = C_{40} = \$1$, $c_n = \$0$, for $n = 1, \ldots, 10$ (first 5 years), $c_n = \$43$, for $n = 11, \ldots, 20$ (next 5 years), $c_n = \$42$, for $n = 21, \ldots, 30$ (next 5 years), and $c_n = \$41$, for $n = 31, \ldots, 40$ (last 5 years). The bond is protected against early redemption for the first 5 years. Set $\kappa_n = 0.1$ and $p_n = \$0$, for $n = 1, \ldots, 40$. The put option is not active. Finally, set $r^c = 0$, $w = 0$, $\sigma = 0.1095$ (per year), and $r^f = 0.0617$ (per year).

We use DP with a grid size of $p = 4000$ to replicate Figure 5 of Brennan and Schwartz (1977). We also report the DP value function of the option-free coupon bond (dashed line) associated to the host bond under interest (solid line) for comparison purposes.

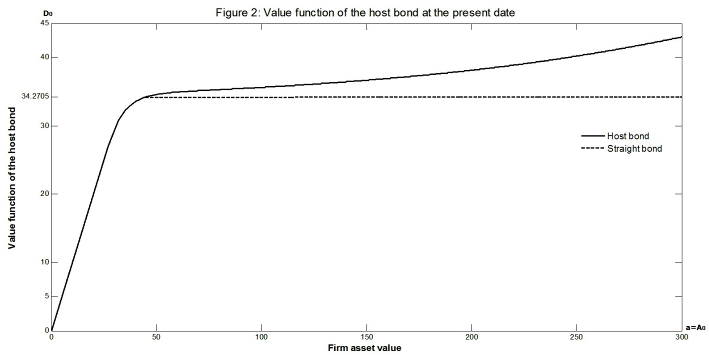

Figure 2

The value functions of the host bond and its option-free counterpart coincide under default. The value of the option-free bond converges to its risk-free counterpart, while the value of the host bond diverges to infiinity, when the asset value tends to infinity. This results from the assumed priority of conversion over redemption.

4.3 Multiple embedded options

The case study is conducted under the lognormal assumption. Consider a balance sheet with $A_0 \in \{25,\ 50,\ 100\}$, $r^c = 0.25$ (per year), $w = 0.25$, $\sigma \in \{0.15, 0.30\}$ (per year), and $r^f = 0.06$ (per year). The host bond is characterized by $N = 5$, $T = t_5 = 5$ (years), $\Delta t = 1$ (year), $P_1 = \cdots = P_4 = \$0$, $P_5 = \$20$, and $C_1 = \cdots = C_5 = \$2$. The call price is $c_n = \$20.5$, for $n = 1, \ldots, 4$, and $c_5 = \$20$, the conversion factor is $\kappa_n = \kappa = 0.20$, for $n = 1, \ldots, 5$, while the put price is $p_n = \$19.5$, for $n = 1, \ldots, 4$, and $p_5 = \$20$.

Table 5 reports DP values of host bonds and their embedded options. CCP stands for the Call, Conversion, and Put option(s). For example, CCP $= 110$ means that the bond hosts the Call and the Conversion options, but not the Put option, while CCP $= 000$ means that the bond is option free. A positive/negative sign of the PVEO indicates a premium/discount value of the host bond with respect to its option-free counterpart.

Figure 3 displays the default/holding/exercise regions at $t_n \in \mathcal{P}$, for $\sigma = 0.3$ (per year). The letters D, P, H, R, F, and C stand for Default, Put, Holding, Redemption, Forced conversion, and Conversion. The letter H between D and P results from our simplified assumption that retraction cannot provoke default. Forced conversion takes place when conversion is not optimal, but it represents a better outcome than redemption for the host-bond holders. Finally, retraction and redemption are not optimal at maturity since $p_5 = c_5 = P_5$.

Redemption tends to decrease the (host) bond value since the call option is at the discretion of the issuer. We find that the (absolute) present value of the call option is higher for low levels of volatility, which is consistent with Kim et al. (1993). The conversion and put options tend to increase the (host) bond value since they are at the discretion of the investor. On the one hand, the put option is

Table 5: DP values of host bonds and their embedded options

		$\sigma = 0.15$			$\sigma = 0.30$		
CCP	A_0	$D_0^h(A_0)$	PVEO	$\mathcal{E}_0(A_0)$	$D_0^h(A_0)$	PVEO	$\mathcal{E}_0(A_0)$
000	25	21.4619	0	4.4057	18.5982	0	6.4489
100	25	20.5118	−0.9501	5.4256	18.3401	−0.2581	6.7149
010	25	21.4623	0.0004	4.4053	18.7135	0.1153	6.3318
001	25	21.4643	0.0024	4.3673	19.0486	0.4504	5.8318
110	25	20.5118	−0.9501	5.4256	18.3402	−0.2580	6.7148
101	25	20.5255	−0.9364	5.3767	18.7874	0.1892	6.1016
011	25	21.4647	0.0028	4.3669	19.1510	0.5528	5.7271
111	25	20.5255	−0.9364	5.3767	18.7874	0.1892	6.1016
000	50	23.1956	0	28.8974	22.4602	0	29.2668
100	50	21.1897	−2.0059	30.9033	21.0977	−1.3625	30.6294
010	50	23.4165	0.2209	28.6765	23.7884	1.3282	27.9387
001	50	23.1957	0.0001	28.8973	22.5951	0.1349	29.1224
110	50	21.1897	−2,0059	30.9033	21.1187	−1.3415	30.6084
101	50	21.1897	−2.0059	30.9033	21.1427	−1.3175	30.5749
011	50	23.4166	0.2210	28.6764	23.9220	1.4618	27.7955
111	50	21.1897	−2.0059	30.9033	21.1637	−1.2965	30.5539
000	100	23.1992	0	78.8965	23.1413	0	78.9216
100	100	21.1897	−2.0095	80.9060	21.1896	−1.9517	80.8733
010	100	28.7948	5.5956	73.3009	30.6152	7.4739	71.4478
001	100	23.1992	0.0000	78.8965	23.1510	0.0097	78.9118
110	100	22.7680	−0.4312	79.3277	23.9162	0.7749	78.1468
101	100	21.1897	−2.0095	80.9060	21.1897	−1.9516	80.8732
011	100	28.7948	5.5956	73.3009	30.6249	7.4836	71.4379
111	100	22.7680	−0.4312	79.3277	23.9162	0.7749	78.1466

dominant for $A_0 = \$25$ and $\sigma = 0.3$ (per year) as retraction is very likely to happen in the near future. On the other hand, the call option is dominant for $A_0 = \$50$ and $\sigma = 0.3$ (per year) as redemption is very likely to happen in the near furture. Finally, for $A_0 = 100$ \$, the conversion option alone is the most valuable with a PV of \$7.4739, but its potentialities are drastically challenged in the presence of the call option even though conversion has priority over redemption.

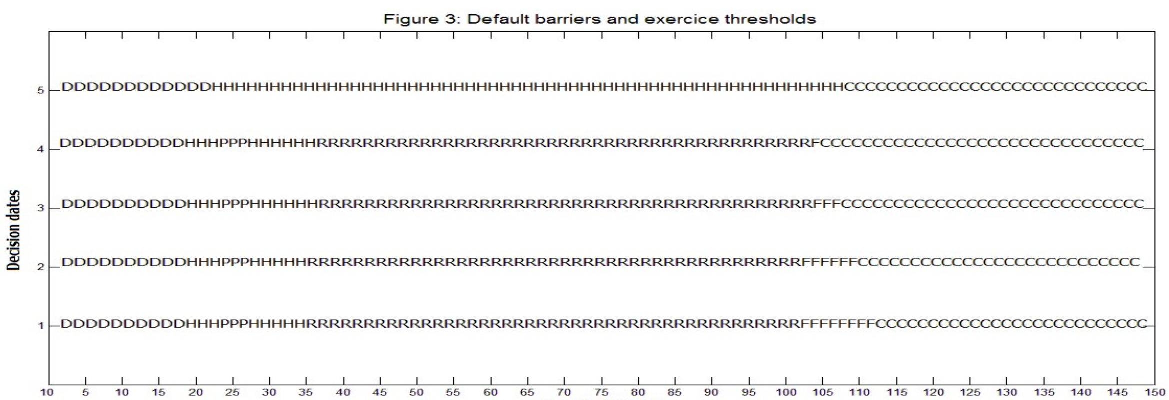

Figure 3

Table 6 displays the term structure of default probabilities of the firm for the case $\sigma = 0.3$ (per year). As expected, redemption drives down the default probabilities, while retraction and conversion have the opposite effect.

Table 6: Default probabilities for the horizon $t_n \in \mathcal{P}$ (in %)

CCP	A_0	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$	$t_4 = 4$	$t_5 = 5$
000	25	9.32	20.57	29.35	36.86	45.97
100	25	9.12	20.42	29.25	36.78	45.91
010	25	9.36	20.61	29.38	36.87	45.99
001	25	12.48	24.40	32.66	39.12	47.64
110	25	9.12	20.42	29.25	36.77	45.91
101	25	12.24	24.20	32.53	39.01	47.56
011	25	12.55	24.44	32.69	39.15	47.66
111	25	12.24	24.20	32.53	39.01	47.56
000	50	0.01	0.53	2.02	4.50	9.44
100	50	0.01	0.52	2.02	4.50	9.44
010	50	0.01	0.53	2.02	4.50	9.44
001	50	0.02	0.70	2.34	4.68	9.55
110	50	0.01	0.52	2.02	4.50	9.44
101	50	0.02	0.70	2.34	4.68	9.55
011	50	0.02	0.70	2.34	4.68	9.56
111	50	0.02	0.70	2.34	4.68	9.55
000	100	0.00	0.00	0.03	0.19	0.84
100	100	0.00	0.00	0.03	0.19	0.84
010	100	0.00	0.00	0.03	0.19	0.84
001	100	0.00	0.00	0.04	0.19	0.84
110	100	0.00	0.00	0.03	0.19	0.84
101	100	0.00	0.00	0.04	0.19	0.84
011	100	0.00	0.00	0.04	0.19	0.84
111	100	0.00	0.00	0.04	0.19	0.84

For $A_0 = \$100$, the entries 000, 100, 010, and 110 of Table 5 are puzzling since the PV of the conversion option alone is \$7.4739, the PV of the call option alone is (\$1.9517), while the PV of the conversion and call options together is 0.7749 \$. The high potentialities of the conversion option are challenged. Where have they gone? Figure 3 shows that a first half of them is collected at t_1, while a second half of them is collected at $t_2, \ldots, t_5 = T$. The second half becomes de facto unattainable in the presence of the call option. We run a new experiment with the same parameters except for the debt maturity date T set at t_1, and we find a PV of the conversion option alone of \$2.7994.

5 Conclusion

We consider an extended structural model with a debt portfolio that contains a host bond with its call, conversion, and put options. We use DP and finite elements to design and solve the model. We show that (early) exercise decisions can be expressed as functions of some redemption, conversion, and retraction thresholds. We replicate a couple of seminal papers, and conduct a numerical investigation that details the effect of each embedded option alone, then their combined effect on the values of the host bond and equity.

Our construction is highly efficient, as it combines DP with local interpolations to approximate the value functions of corporate securities. It is also highly flexible in that it accomodates alternative Lévy processes, various tangible and intangible corporate securities, realistic debt payment schedules, multiple seniority classes, and the call, conversion, and put options embedded in corporate bonds.

Promizing research avenues consist of extending our DP approach to

1. additional intangible assets, such as reorganization costs and illiquity costs, to design and evaluate strategic poison puts;
2. alternative Lévy dynamics of the asset value and the risk-free rate in order to design and evaluate exchangeable bonds.

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