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Feedback Stackelberg-Nash equilibria in difference games with quasi-hierarchical interactions and inequality constraints

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Abstract : In this work, we study a class of two-player deterministic finite-horizon difference games with coupled inequality constraints, where both players have two types of decision variables. In one type of decision variables players interact sequentially whereas in the other type they interact simultaneously. We refer to this class of games as quasi-hierarchical dynamic games and define a solution concept called feedback Stackelberg-Nash (FSN) equilibrium. Under separability assumption of cost functions, we provide a recursive formulation of the FSN solutions using an approach similar to dynamic programming. Furthermore, we show that the FSN solution of this class of constrained games can be obtained from the parametric feedback Stackelberg solution of an associated unconstrained parametric game involving only sequential interactions, with a specific choice of the parameters that satisfy some implicit complementarity conditions. For the linear-quadratic case, we numerically obtain the FSN solutions by reformulating these implicit complementarity conditions as a single large-scale linear-complementarity problem. Finally, we illustrate our results using a dynamic duopoly game with production constraints.

Keywords: Difference games, feedback Stackelberg-Nash equilibrium, coupled inequality constraints, complementarity problem

1 Introduction

Dynamic game theory (DGT) provides a mathematical framework to model multi-agent decision processes that evolve over time. In contrast to static games, where players/agents act only once, dynamic games involve multiple (or even continuous) sequential or simultaneous decisions over a given, or endogenously determined, planning horizon. This temporal aspect comes with strategic complexities, requiring players to not only consider their immediate choices but also anticipate and respond to future decisions made by others over time. The practical utility of DGT is well evident from its successful applications across diverse fields such as engineering, management science, and economics, where dynamic multi-agent decision problems naturally arise (see [2, 5, 15, 16, 20, 23] and the handbook on DGT [3]). In particular, in engineering DGT has been applied to address problems in, e.g., cyber-physical systems [46], communication and networking [7, 44], autonomous vehicles [17], and smart grids [25].

The two main solution concepts in noncooperative dynamic games are Nash and Stackelberg equilibria. One main difference between the two concepts is the information available to a player when making her decision. In a Nash game [30], the information is imperfect, that is, each player makes her strategic choice without knowing the decisions made by the others. In a Stackelberg game [39], the information is perfect, that is, each player knows the opponent's last move when choosing her decision. Put differently, whereas the mode of play is simultaneous in a Nash game, it is sequential, or hierarchical, in two-player Stackelberg game, where one player acts first (the leader), and the other one (the follower) best replies to the leader's decision. Anticipating the follower's response, the leader chooses a strategy that optimizes her performance index. Nash equilibrium solution has been retained in various engineering applications such as smart grid [25], communication and networking [44], collision avoidance [29], and formation control [19]. Stackelberg equilibrium has been implemented in, e.g., communication networks [7], demand response management in smart grid [25], supply chains and marketing channels [13, 21, 24], and traffic routing [18].

Two features are often observed in the literature in dynamic games, both in discrete and continuous time. Firstly, the models assume away constraints linking the control (strategies, decisions) and state variables. To illustrate, theorems giving the conditions for existence and uniqueness of Nash and Stackelberg equilibria are typically stated without considering coupling between the players' decision sets, and mixed control-state constraints; see, e.g., the textbooks [2, 15, 16, 20]. Clearly, practical considerations such as saturation constraints, bandwidth limitations, production capacity, budget constraints, and limits on pollution emissions, come into play in real-world multi-agent decision situations. When integrated into a dynamic game, these factors introduce equality and inequality constraints in the model, which couples the players' decision sets at each stage of the game. In [31] and [32], the authors analyzed, respectively, open loop and feedback Nash equilibria for linear-quadratic (LQ) games under private type constraints. Recently, [26] characterized open-loop Nash equilibrium strategies under coupled constraints, and [27] studied the open-loop Nash equilibrium in the mean-field-type games in the presence of deterministic coupled constraints. To the best of our knowledge, the hierarchical counterparts to the setups in [26, 27, 31, 32] have not yet been considered.

Secondly, with a few exceptions [4, 8, 9, 11, 22, 42], the mode of play in (typically) two-player games is either simultaneous or sequential. In [11], the players assume in turn the roles of leader and follower when setting their advertising budgets (control variables), which affect their market shares (state variables) and profits. In [4, 8, 9, 22, 42], the mode of play is referred to as mixed-leadership, where each of the two players make decisions on two variables. First, they select simultaneously the levels of one type of variables, and second, they react, also simultaneously, by choosing the levels of the other variable. Therefore, we have two Nash games and one Stackelberg game. Mixed-leadership games have been applied to cooperative advertising [8] and innovation and pricing decisions [22].

In this paper, we are interested in a class of two-player finite-horizon dynamic games, where at any period k, each player i has two types of decision variables u_k^i and v_k^i. At any k, with the exception of the last period, the decision process is decomposed into three stages. In the first stage, the leader announces an action u_k^1 and in the second stage, the follower responds by choosing u_k^2. In the third stage, the two players choose simultaneously v_k^1 and v_k^2. At terminal instant K, the players only interact simultaneously in their choices of v_K^1 and v_K^2. From this description, it is clear that our class of games, to which we shall refer as *quasi-hierarchical* dynamic games, differ from the class of mixed-leadership games by having a sequential interaction followed by a simultaneous one instead of having two simultaneous interactions. We illustrate our setup with two simple generic examples.

As a first example, we consider a supply chain formed of a manufacturer and a retailer and let the decision process at each period be decomposed into three stages. In the first stage, the manufacturer (the leader) announces the wholesale price of the product to the retailer (the follower) who reacts in a second stage by choosing the price to consumers. In the third stage, both players choose simultaneously some demand-enhancing activities, e.g., national brand advertising by the manufacturer and local advertising by the retailer. Note that advertising expenditures are typically upper bounded by the available budget.

In the previous example, we have a so-called *vertical* strategic interaction between the players, that is, a same product is sold by the manufacturer to the retailer who sells it to consumers. As a second example, we consider a foreign firm and a local company competing in the same market by offering two partially substitutable products, and the demand to each firm depends on both competitors' prices. Here, the strategic interaction is *horizontal*.[1] As before, the decision process at any period is decomposed into three stages. In the first two stages, the firms adjust sequentially their production capacities by adding, or decommissioning, some equipment, and in the third stage they compete in prices. We suppose that the foreign firm announces first its global investment strategy, including in the market of interest, which gives the local firm the opportunity to observe the capacity adjustment made by the foreign firm before adjusting its own in the second stage. Therefore, in any time period, the investment decisions are sequential, with the foreign firm acting as leader and the local firm as follower, while pricing decisions are next made simultaneously by both firms. In this example, the quantity produced must be non-negative and is upper bounded by the production capacity.

Inspired by the two illustrative examples given above, our dual objective is to characterize equilibrium strategies for the class of two-player quasi-hierarchical dynamic games, with mixed coupled inequality constraints, and to provide a method for computing them.

1.1 Related literature

Nash equilibrium was first introduced in dynamic games setting in [40, 41]. In particular, [40] emphasized the significance of available information to the players during their decision-making process in dynamic games, noting how it leads to different types of Nash equilibria. Similarly, [36, 38] extended the notion of Stackelberg solution to multi-period settings using a control-theory framework. Additionally, in [37], the authors also introduced the notion of a feedback Stackelberg solution where the leader enforces stage-wise her policy choices on the follower rather than globally. This solution concept requires the players to know the current state of the game at each time instant. The derivation of feedback Stackelberg solution involves a backward recursion [2], similar to dynamic programming, where at each step of the recursion, the Stackelberg solution of a static game is determined. In scenarios where the leader possesses dynamic information and can announce in advance her policy for the entire duration of the game [10], the computation of Stackelberg solution, while conceptually well-defined, becomes challenging. This difficulty arises because the underlying optimization problems involve the policy spaces of both players, with reaction sets typically being of infinite dimension. In [1] and [6],

[1] The meaning and impact of cooperation are different in these two types of interactions. Whereas coordination of players' strategies in a vertical channel is socially desirable because it leads to both higher profits for firms and higher consumer surplus, coordination between competing firms is synonymous to collusion between firms, which is detrimental to consumer surplus and total welfare.

indirect methods using an incentive approach have been proposed for the derivation of such global Stackelberg solutions, in discrete and continuous time setting, respectively.

In [4], the authors introduced the class of mixed-leadership games under open-loop information structure and used the maximum principle to characterize an open-loop Stackelberg solution, which led to a set of algebraic equations and differential equations with mixed-boundary conditions. In particular, for LQ differential games they showed that an associated coupled Riccati equations with mixed-boundary conditions must be solved to express the Stackelberg solution in terms of the system state. The existence of unique open-loop Stackelberg equilibrium for two player LQ differential games with mixed leadership was investigated in [9]. The authors also provided sufficient conditions for the existence and uniqueness of solutions to the associated coupled Riccati equations with mixed-boundary conditions. Stochastic mixed-leadership games under feedback information structure are analyzed in [8], where the diffusion term is assumed to be independent of players controls. In [22], a mixed-leadership game under feedback information structure is studied with state and control dependent diffusion term in the state dynamic.

In DGT framework, the simultaneous interaction has been studied in the presence of dynamic mixed state control constraints. In particular, the existence of constrained open-loop and feedback Nash equilibria for a specific class of LQ difference games with affine inequality constraints are analyzed in [31] and [32], respectively. These works considered two types of control variables: one influencing the state evolution, and another only affecting the constraints. Therefore, the control variables associated with the first type are not coupled. Recently, [26] and [27] characterized open-loop Nash equilibria in games with coupled constraints, and in mean-field-type LQ games, respectively. In [28], the necessary and sufficient conditions for the existence of open-loop Stackelberg solution in two-player LQ difference games with constraints was studied. However, their analysis was restricted to a simpler case where only the leader had linear state control inequality constraints with the follower having no constraint. Global Stackelberg solution for stochastic games under adapted open-loop and closed-loop memoryless information structure with convex control constraints were studied in [45]. In [42], mixed-leadership games with input constraints were considered. Their analysis was restricted to the case where only the decision variables, for which players act as leader, were constrained to be in a closed convex set. However, the other decision variables, for which players act as follower, were free.

1.2 Contribution

The contribution of our paper to the literature in dynamic games is three-fold. First, we introduce the new class of quasi-hierarchical dynamic games, which we believe has practical relevance as illustrated in the supply chain and duopoly examples. One difference with respect to its closest class of mixed-leadership games [4, 8, 9, 22, 42], lies in the information structure and type of interactions between the players. Instead of having two successive simultaneous interactions, here we have one sequential interaction followed by a simultaneous one. In short, we have a different model of strategic interactions, and consequently a different solution. Second, we define an equilibrium concept for this class of games, i.e., the feedback Stackelberg-Nash (FSN) equilibrium, and provide sufficient conditions for its existence. Finally, we provide a computational approach of the FSN equilibrium. We show that a solution to the original constrained difference game can be obtained from a parametric feedback Stackelberg solution of an associated unconstrained parametric game involving only sequential interactions, with a specific choice of the parameters that satisfy some implicit complementarity conditions. Further, we show that the FSN equilibrium of a linear quadratic game can be obtained by reformulating these complementarity conditions as a single large scale linear complementarity problem.

The rest of the paper is organized as follows. In Section 2, we introduce the finite-horizon nonzero-sum difference games with coupled inequality constraints involving two types of decision variables and also describe the quasi-hierarchical interaction. We define the feedback Stackelberg-Nash (FSN) solution in Section 3. In Section 4.3, we present a sufficient condition for the existence of FSN solution in these games. Next, in Section 4.4, we show that the FSN solution can be obtained from the parametric feedback Stackelberg solution of an associated unconstrained parametric game involving only sequential decision variables, with a specific choice of the parameters satisfying some implicit complementarity conditions. We specialize these results to a linear-quadratic setting involving affine inequality constraints in Section 5. In Section 6 we illustrate our results using a simple dynamic duopoly game between a foreign firm and a local firm. Finally, conclusions are provided in Section 7.

Notation: We denote real numbers as $\mathbb{R}$, non-negative real numbers as $\mathbb{R}_+$, n-dimensional Euclidean space as $\mathbb{R}^n$, and $n \times m$ real matrices as $\mathbb{R}^{n\times m}$. For any matrix $A \in \mathbb{R}^{n\times m}$, its transpose is denoted $A' \in \mathbb{R}^{m\times n}$. The identity matrix and matrix with all zero entries are denoted by $\mathbf{I}$ and $\mathbf{0}$, with dimensions inferred from context. Let $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}^n$, where $n = n_1 + n_2 + \cdots + n_K$. We represent $[A]_{ij}$ as the $n_i \times n_j$ sub-matrix with row indices n_i and column indices n_j, and $[a]_i$ as the $n_i \times 1$ sub-vector with index n_i. The column vector $[v_1', \cdots, v_n']'$ is denoted by $\mathrm{col}(v_1, \cdots, v_n)$ or more compactly as $\mathrm{col}(v_k)k = 1^n$. The block diagonal matrix formed by taking matrices $M_1, M_2, \cdots, M_K$ as diagonal elements is denoted by $\oplus_{k=1}^K M_k$. Vectors $x, y \in \mathbb{R}^n$ are complementary if $x \geq 0$, $y \geq 0$, and $x'y = 0$, denoted as $0 \leq x \perp y \geq 0$. The composition of functions $f(\cdot) : \mathbb{R}^l \to \mathbb{R}^m$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^l$ is denoted as $(f \circ g)(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$.

2 Preliminaries

2.1 Dynamic game with inequality constraints

In this section, we introduce a class of two-person discrete-time nonzero-sum finite-horizon dynamic game involving inequality constraints (CNZDG). Let $\{1, 2\}$ denote the set of players and $\mathsf{K} = \{0, 1, ..., K\}$ denote the set of decision instants or time periods. We also define the two set $\mathsf{K}_l := \mathsf{K} \setminus \{K\}$ and $\mathsf{K}_r := \mathsf{K} \setminus \{0\}$. At each time instant $k \in \mathsf{K}_l$, player $i \in \{1, 2\}$ chooses an action $u_k^i \in \mathsf{U}_k^i$, where $\mathsf{U}_k^i \subset \mathbb{R}^{m_i}$ denotes the admissible set of actions for player i. Players, through their actions, influence the evolution of state variable $x_k \in \mathbb{R}^n$ according to the following discrete-time dynamics:

$$x_{k+1} = f_k(x_k, u_k^1, u_k^2), \quad k \in \mathsf{K}_l, \tag{1}$$

where $f_k : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^n$ and the initial state $x_0 \in \mathbb{R}^n$ is given. We assume that at each time instant $k \in \mathsf{K}$ both players are endowed with additional decision variables $v_k^i \in \mathbb{R}^{s_i},\ i \in \{1, 2\}$, which do not enter the dynamics directly, but only influence the player's decision making process at every time instant $k \in \mathsf{K}$ in the form of the following mixed inequality constraints:

$$h_k^i(x_k, v_k^1, v_k^2) \geq 0,\ v_k^i \geq 0,\ k \in \mathsf{K},\ i \in \{1, 2\}, \tag{2}$$

where $h_k^i : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}_+^{c_i}$. Observe that the constraints (2) are coupled, that is, at each stage $k \in \mathsf{K}$, player 1's control action v_k^1 is related to player 2's control action v_k^2 and vice versa. Using the above constraints (2) of both players, we define the joint admissible action set $\mathsf{V}_k(x_k)$ at stage $k \in \mathsf{K}$. To this end, we first introduce the reachable set $\mathsf{X}_k \subset \mathbb{R}^n$, that is, the set of all state variables at stage k that are reachable when players $i \in \{1, 2\}$ use some arbitrary combinations of admissible actions $u_\tau^i \in \mathsf{U}_\tau^i, \tau = 0, \cdots, k-1$ in the upstream stages. For any $x_k \in \mathsf{X}_k$, the joint admissible action set $\mathsf{V}_k(x_k)$ at stage $k \in \mathsf{K}$ is defined as

$$\mathsf{V}_k(x_k) := \{(v_k^1, v_k^2) \in \mathbb{R}^s \mid h_k^i(x_k, v_k^1, v_k^2) \geq 0,\ v_k^i \geq 0,\ i \in \{1, 2\}\}.$$

The admissible action set of player $i \in \{1, 2\}$ for decision variable $v_k^i \in \mathbb{R}^{s_i}$, for a given $x_k \in \mathsf{X}_k$ and action v_k^j of other player $j \in \{1, 2\}, i \neq j$, is defined as follows:

$$\mathsf{V}_k^i(v_k^j) := \{v_k^i \in \mathbb{R}^{s_i} \mid (v_k^1, v_k^2) \in \mathsf{V}_k(x_k)\}. \tag{3}$$

We denote the joint actions of players by $\mathsf{u}_k := \mathrm{col}(u_k^1, u_k^2),\ k \in \mathsf{K}_l$ and $\mathsf{v}_k := \mathrm{col}(v_k^1, v_k^2),\ k \in \mathsf{K}$. The collection or profile of actions of player $i \in \{1, 2\}$ is denoted by $(\tilde{\mathsf{u}}^i, \tilde{\mathsf{v}}^i)$, where $\tilde{\mathsf{u}}^i := \mathrm{col}(u_k^i)_{k=0}^{K-1}$

and $\tilde{\mathsf{v}}^i := \mathrm{col}(v_k^i)_{k=0}^K$. The corresponding strategy set of player i is $\prod_{k=0}^{K-1} \mathsf{U}_k^i \times \prod_{k=0}^{K} \mathsf{V}_k^i$. Player i ($i \in \{1,2\}$) minimizes the following stage-additive cost functional:

$$J_i(x_0, (\tilde{\mathsf{u}}^1, \tilde{\mathsf{v}}^1), (\tilde{\mathsf{u}}^2, \tilde{\mathsf{v}}^2)) = g_K^i(x_K, \mathsf{v}_K) + \sum_{k=0}^{K-1} g_k^i\left(x_k, \mathsf{u}_k, \mathsf{v}_k\right), \tag{4}$$

where $g_k^i : \mathbb{R}^n \times \mathbb{R}^{m_1+m_2} \times \mathbb{R}^{s_1+s_2} \to \mathbb{R}$, $k \in \mathsf{K}_l$ and $g_K^i : \mathbb{R}^n \times \mathbb{R}^{s_1+s_2} \to \mathbb{R}$ denote the instantaneous and terminal cost functions of player i, respectively.

Remark 1. The dynamic game described by (1)–(4) was first studied in [31], [32], and [33] in a linear-quadratic setting. In this class of games, the decision variables u_{k-1} at time instant $k-1$ influence the state variable x_k at instant k. On the other hand, the decision variables v_k at instant k are affected by the state variable x_k of the same instant k through the inequality constraints (2). In other words, the decision variables u_k^i, $i \in \{1,2\}$, influence the constraints (2) indirectly through the state Equation (1).

We make the following assumptions related to the dynamic game (1)–(4):

Assumption 1.

(i) The admissible action sets $\mathsf{U}_k^i \subset \mathbb{R}^{m_i}$ for $k \in \mathsf{K}_l$, $i \in \{1,2\}$ are such that the joint action sets $\mathsf{V}_k(x_k)$ for all $k \in \mathsf{K}$ are nonempty, convex, closed and bounded.

(ii) The functions $\{f_k,\ k \in \mathsf{K}_l,\ h_k^i,\ g_k^i,\ k \in \mathsf{K},\ i \in \{1,2\}\}$ are continuously differentiable in their arguments.

(iii) The matrices $\left\{\frac{\partial h_k^i}{\partial v_k^i},\ k \in \mathsf{K},\ i \in \{1,2\}\right\}$ are full rank for each $\mathsf{v}_k \in \mathsf{V}_k(x_k)$ and $x_k \in \mathsf{X}_k$.

We recall from Remark 1 that the control actions from the sets $\mathsf{U}_{k-1}^i, i \in \{1,2\}$ influence the state x_k at time k. So, to ensure the feasibility of the joint action sets $\mathsf{V}(x_k)$, Assumption 1.(i) is required. Assumption 1.(iii) ensures that the constraint qualification conditions hold; see also [12] for other formulations of constraint qualifications.

2.2 Quasi-hierarchical interaction and information structure

In this paper, we analyze a quasi-hierarchical interaction between the players, in the two types of decision variables, as described in the following definition:

Definition 1 (Quasi-hierarchical interaction)**.** The leader and follower are denoted by the labels 1 and 2, respectively. Between any two time instants k and $k+1$, with $k \in \mathsf{K}_l$, the players interact in three stages. In the first stage, the leader announces her action u_k^1 and in the second stage, the follower responds by announcing her action u_k^2. In the third stage, the players choose simultaneously their decisions v_k^1 and v_k^2. At the terminal instant K, the players choose simultaneously the decision variables v_K^1 and v_K^2.

The decision process between the time instants k and $k+1$ is illustrated in Figure 1. In the sequential interaction after the leader announces her action $u_k^1 \in \mathsf{U}_k^1$ and the follower gives her response $u_k^2 \in \mathsf{U}_k^2$, the state variable x_{k+1} is determined. The state variable x_k influences the joint actions set $\mathsf{V}_k(x_k)$ of the players when they choose actions (v_k^1, v_k^2) simultaneously in the third stage. Before the start of the next period $k+1$, player $i \in \{1,2\}$ incurs a cost $g_k^i(x_k, \mathsf{u}_k, \mathsf{v}_k)$. At terminal stage K, the two players interact simultaneously and player $i \in \{1,2\}$ incurs a cost $g_K^i(x_K, \mathsf{v}_K)$.

In the presence of constraints, [31, 32] introduce constrained information structure both in the open-loop and feedback forms, and demonstrate that this information structure leads to a semi-analytic characterization of Nash equilibrium strategies. Motivated by this aspect, in this paper, we assume constrained feedback information structure, which is described as follows; see also [32, Section III.A].

Definition 2 (Constrained feedback information structure)**.** The control action (u_k^i, v_k^i) of player i ($i \in \{1,2\}$) at stage k is given by $u_k^i := \gamma_k^i(x_k) \in \mathsf{U}_k^i$ and $v_k^i \in \mathsf{V}_k^i(v_k^{-i})$, where $\gamma_k^i : \mathbb{R}^n \to \mathsf{U}_k^i$ is a measurable mapping and the set of all such mappings is denoted by Γ_k^i.

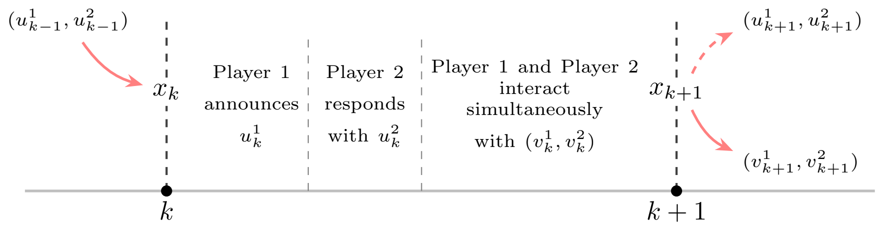

Figure 1: Three step decision process involving sequential and simultaneous interactions between the time instants k and $k+1$.

3 Feedback Stackelberg-Nash solution

First, we introduce some additional notations. Let $\psi^i_k := (\gamma^i_k, v^i_k) \in \Psi^i_k$ denote the mapping associated with the joint action (u^i_k, v^i_k), $i \in \{1,2\}$, where we represent the joint action space as Ψ^i_k, i.e., $\forall k \in \mathsf{K}_l, \Psi^i_k := \Gamma^i_k \times V^i_k$ and $\Psi^i_K := V^i_K$. Consequently, we have $(u^i_k, v^i_k) = (\gamma^i_k(x_k), v^i_k) = \psi^i_k$ for $k \in \mathsf{K}_l$ and $\psi^i_K := v^i_K$ at the terminal time. We denote the strategy of player i by $\psi^i := ((\gamma^i_k(x_k), v^i_k)_{k\in\mathsf{K}_l}, v^i_K)$ and the joint strategy (ψ^1, ψ^2) by ψ. The strategy space for player i is denoted by $\Psi^i := \prod_{k=0}^{K} \Psi^i_k$. For any $a, b \in \mathsf{K}$, with $0 \le a \le b \le K$, we denote the collection of actions for time periods from a to b (including both a and b) by $\psi^i_{[a,b]} := (\psi^i_a, \cdots, \psi^i_b)$ and the corresponding admissible sets by $\Psi^i_{[a,b]} := \prod_{k=a}^{b} \Psi^i_k$. Using the above notations, we define the FSN solution as follows:

Definition 3 (Feedback Stackelberg-Nash solution)**.** A pair $(\psi^{1\star}, \psi^{2\star}) \in (\Psi^1, \Psi^2)$ constitutes a *feedback Stackelberg-Nash solution* for CNZDG if the following conditions are satisfied:

1. For all $\psi^i_{[0,K-1]} \in \Psi^i_{[0,K-1]}$, $i \in \{1,2\}$, $(\psi^{1\star}_K, \psi^{2\star}_K) \in \Psi^1_K \times \Psi^2_K$ satisfy

$$J_1\big(x_0, (\psi^1_{[0,K-1]}, \psi^{1\star}_K), (\psi^2_{[0,K-1]}, \psi^{2\star}_K)\big) \le J_1\big(x_0, (\psi^1_{[0,K-1]}, \psi^1_K), (\psi^2_{[0,K-1]}, \psi^{2\star}_K)\big), \ \forall \psi^1_K \in \Psi^1_K, \tag{5a}$$

$$J_2\big(x_0, (\psi^1_{[0,K-1]}, \psi^{1\star}_K), (\psi^2_{[0,K-1]}, \psi^{2\star}_K)\big) \le J_2\big(x_0, (\psi^1_{[0,K-1]}, \psi^{1\star}_K), (\psi^2_{[0,K-1]}, \psi^2_K)\big), \ \forall \psi^2_K \in \Psi^2_K. \tag{5b}$$

2. Recursively for $k = K-1, \cdots, 0$ and for all $\psi^i_{[0,k-1]} \in \Psi^i_{[0,k-1]}$, $i \in \{1,2\}$, in the decision process between time instants k and $k+1$,

   (a) For all $(\gamma^1_k, \gamma^2_k) \in \Gamma^1_k \times \Gamma^2_k$, $(v^{1\star}_k, v^{2\star}_k) \in \mathsf{V}_k$ satisfy in the third stage the following inequalities:

$$\begin{aligned} &J_1\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\gamma^2_k, v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big) \\ &\le J_1\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^1_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\gamma^2_k, v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big), \ \forall v^1_k \in V^1_k(v^{2\star}_k), \end{aligned} \tag{5c}$$

$$\begin{aligned} &J_2\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\gamma^2_k, v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big) \\ &\le J_2\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\gamma^2_k, v^2_k), \psi^{2\star}_{[k+1,K]})\big), \ \forall v^2_k \in V^2_k(v^{1\star}_k). \end{aligned} \tag{5d}$$

   (b) For every $\gamma^1_k \in \Gamma^1_k$, there exists a unique map $\mathsf{R}^2_k : \Gamma^1_k \to \Gamma^2_k$ such that the following inequality is satisfied in the second stage:

$$\begin{aligned} &J_2\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\mathsf{R}^2_k(\gamma^1_k), v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big) \\ &\le J_2\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\gamma^2_k, v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big), \ \forall \gamma^2_k \in \Gamma^2_k. \end{aligned} \tag{5e}$$

   (c) In the first stage, $\gamma^{1\star}_k \in \Gamma^1_k$ satisfies the inequality

$$\begin{aligned} &J_1\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^{1\star}_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\mathsf{R}^2_k(\gamma^{1\star}_k), v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big) \\ &\le J_1\big(x_0, (\psi^1_{[0,k-1]}, (\gamma^1_k, v^{1\star}_k), \psi^{1\star}_{[k+1,K]}), (\psi^2_{[0,k-1]}, (\mathsf{R}^2_k(\gamma^1_k), v^{2\star}_k), \psi^{2\star}_{[k+1,K]})\big), \ \forall \gamma^1_k \in \Gamma^1_k, \end{aligned} \tag{5f}$$

   with $\gamma^{2\star}_k = \mathsf{R}^2_k(\gamma^{1\star}_k)$.

The feedback Stackelberg-Nash equilibrium costs incurred by the leader and the follower are given by $J_1(x_0, \psi^{1\star}, \psi^{2\star})$ and $J_2(x_0, \psi^{1\star}, \psi^{2\star})$, respectively.

In Definition 3, the conditions (5) characterize the FSN solution using dynamic programming. As a result, these conditions decompose the computation of FSN solution as solving $K+1$ static game problems backward in time. Since (v_K^1, v_K^2) are the only decision variables involved at the terminal period K, and the interaction in these decision variables is simultaneous, the outcome is a Nash equilibrium $(\psi_K^{1\star}, \psi_K^{2\star})$. More specifically, the condition (5a)–(5b) implies that whatever admissible strategies used by the players to reach period K, that is, for any arbitrary choice of admissible strategies $\psi^i_{[0,K-1]} \in \Psi^i_{[0,K-1]}$, the decisions specified by FSN solution $(\psi_K^{1\star}, \psi_K^{2\star})$ satisfy conditions (5a)–(5b) at K. Next, for all other time periods $k \in \mathsf{K}_l$, the players interact sequentially using (u_k^1, u_k^2) and simultaneously using (v_k^1, v_k^2). The FSN solutions are computed during the time periods $k \in \mathsf{K}_l$, for any arbitrary choice of admissible upstream strategies $(\psi^1_{[0,k-1]}, \psi^2_{[0,k-1]})$, by fixing the downstream decisions at $(\psi^{1\star}_{[k+1,K]}, \psi^{2\star}_{[k+1,K]})$. Between the time periods k and $k+1$, the players interact according to the three-stage decision process pictured in Figure 1. Again, using dynamic programming principle, starting from the third stage, the conditions (5c)–(5d) imply that the interaction is simultaneous and $(v_k^{1\star}, v_k^{2\star})$ is the corresponding Nash outcome. At the second stage, fixing the third stage decisions at $(v_k^{1\star}, v_k^{2\star})$, (5e) implies that the follower gives best response, denoted by $\mathsf{R}_k^2(\gamma_k^1)$, for every leader's announcement $\gamma_k^1 \in \Gamma_k^1$. At the first stage, (5f) implies that the leader minimizes her cost, by taking the follower's best response obtained at stage two, to obtain her FSN Stackelberg strategy $\gamma_k^{1\star} \in \Gamma_k^1$.

Remark 2. If the decision variables (v_k^1, v_k^2), the associated constraints (2), and the corresponding terms in the objectives (4), are absent in the model, then the FSN solution coincides with the canonical feedback-Stackelberg solution; see [2, Definition 3.29].

Remark 3. In Definition 3, it is required that the follower's rational reaction set $\mathsf{R}_k^2(\gamma_k^1)$ is a singleton for every leader's announcement $\gamma_k^1 \in \Gamma_k^1$ at all periods $k \in \mathsf{K}_l$. In the subgame starting at stage k, if $\mathsf{R}_k^2(\gamma_k^1)$ is not a singleton, then the leader's cost depends on the strategy used by the follower from $\mathsf{R}_k^2(\gamma_k^1)$. The leader can secure her cost against multiple follower's optimal responses from $\mathsf{R}_k^2(\gamma_k^1)$ by considering the follower's response that gives her the worst cost. When $\mathsf{R}_k^2(\gamma_k^1)$ is not a singleton, the leader cannot enforce her strategy on the follower, which creates an important difficulty in determining a Stackelberg equilibrium. This explains why the literature typically assumes that $\mathsf{R}_k^2(\gamma_k^1)$ is a singleton [2, Theorem 7.1 and Theorem 7.2].

Remark 4. In Definition 3, the recursive formulation of FSN solution leads to a static game problem at every instant k, where the downstream decisions are fixed at FSN solutions for stages $k+1, \cdots, K$. During this recursive procedure if there exist, at any time instant k, more than one solution $(v_k^{1\star}, v_k^{2\star})$, then the upstream static games need to be solved for each one of these solutions to obtain the complete set of FSN solutions.

4 Recursive formulation of feedback Stackelberg-Nash solution

In this section, we provide a recursive formulation of the FSN solution. To this end, we make the following assumption on the players' cost functions:

Assumption 2. For all $k \in \mathsf{K}_l$ and $i \in \{1,2\}$ the players' cost functions have the following separable structure:

$$g_k^i(x_k, \mathsf{u}_k, \mathsf{v}_k) = g_{u,k}^i(x_k, \mathsf{u}_k) + g_{v,k}^i(x_k, \mathsf{v}_k). \tag{6}$$

This assumption excludes having a cross term between the sequential and simultaneous decision variables in the players' objective. The following example illustrates the complications induced by the presence of such cross term:

Example 1. Consider a scalar two-period game with state dynamics given by $x_1 = x_0 - (u_0^1 + u_0^2)$ and the initial state $x_0 \in \mathbb{R}$. The leader minimizes the cost function $J_1 = 4(x_1)^2 + 2(u_0^1)^2 + (u_0^2)^2 + (v_0^1)^2 + v_0^1 v_0^2$ subject to the constraints $x_0 + v_0^1 \geq 0$, $v_0^1 \geq 0$. Similarly, the follower minimizes the cost $J_2 = (x_1)^2 + (u_0^2)^2 + u_0^2 v_0^2 + (v_0^2)^2 + v_0^1 v_0^2$ subject to the constraints $x_0 + 2v_0^2 \geq 0$ and $v_0^2 \geq 0$. Since the players have decision variables only for stage 0, we consider the three stage decision process for this time period. For any admissible sequential actions u_0^1 and u_0^2 of leader and follower, respectively, the Nash (simultaneous) solution at stage 0 will be given by the following optimization problems:

$$\min_{v_0^1 \geq 0} \left\{4(x_1)^2 + 2(u_0^1)^2 + (u_0^2)^2 + (v_0^1)^2 + v_0^1 v_0^{2\star}\right\} \quad \text{subject to } x_0 + v_0^1 \geq 0.$$

$$\min_{v_0^2 \geq 0} \left\{(x_1)^2 + (u_0^2)^2 + u_0^2 v_0^2 + (v_0^2)^2 + v_0^{1\star} v_0^2\right\} \quad \text{subject to } x_0 + 2v_0^2 \geq 0.$$

Using the relation $x_1 = x_0 - (u_0^1 + u_0^2)$, the associated Lagrangians can be written as $4(x_0-(u_0^1+u_0^2))^2 + 2(u_0^1)^2+(u_0^2)^2+(v_0^1)^2+v_0^1v_0^{2\star}-\mu_0^1(x_0+v_0^1)$ and $(x_0-(u_0^1+u_0^2))^2+(u_0^2)^2+u_0^2v_0^2+(v_0^2)^2+v_0^{1\star}v_0^2-\mu_0^2(x_0+2v_0^2)$. Here, μ_0^1 and μ_0^1 are the Lagrange multipliers associated with the inequality constraints $x_0 + v_0^1 \geq 0$ and $x_0 + 2v_0^2 \geq 0$, respectively. The associated KKT conditions are

$$\left.\begin{aligned} &0 \leq 2v_0^{1\star} + v_0^{2\star} - \mu_0^{1\star} \perp v_0^{1\star} \geq 0, \\ &0 \leq u_0^2 + 2v_0^{2\star} + v_0^{1\star} - 2\mu_0^{2\star} \perp v_0^{2\star} \geq 0, \\ &0 \leq x_0 + v_0^{1\star} \perp \mu_0^{1\star} \geq 0, 0 \leq x_0 + 2v_0^{2\star} \perp \mu_0^{2\star} \geq 0 \end{aligned}\right\} \tag{7}$$

Next, at stage two, the optimal response of the follower for any leader's admissible action u_0^1, upon fixing the third stage decisions at $(v_0^{1\star}, v_0^{2\star})$, is obtained by solving the optimization problem (5e), which after substituting for $x_1 = x_0 - (u_0^1 + u_0^2)$ is given by

$$\min_{u_0^2} \left\{(x_0 - (u_0^1 + u_0^2))^2 + (u_0^2 + v_0^{2\star})^2 - u_0^2 v_0^{2\star} + v_0^{1\star} v_0^{2\star}\right\} \quad \text{subject to } (7).$$

Due to presence of the cross term $u_0^2 v_0^2$ between the sequential and simultaneous decision variables in the follower's objective, her optimization problem is a mathematical programming problem with complementarity constraint (MPCC). In general, MPCC are extremely hard to solve because of the non-convexity and non-smoothness of the feasible set. Also, the lack of constraints qualification in every feasible point makes them intractable; see [35, 43]. □

Next, using Definition 3, we develop a recursive formulation of FSN solution. To this end, we make use of the reachable set $\mathsf{X}_k \subset \mathbb{R}^n$, which is the set of all state variables at stage k that are reachable when players use admissible strategies in $\Psi^1_{[0,k-1]} \times \Psi^2_{[0,k-1]}$. Let the FSN strategies of the players be denote by $\{\psi^{i\star} \equiv (\{(\gamma_k^{i\star}(x_k), v_k^{i\star})\}_{k\in\mathsf{K}_l}, v_K^{i\star}),\ i \in \{1,2\}\}$. Recall that the FSN solution characterized in Definition 3 at time k leads to a static game at the same time where the down-stream decisions from instants $k+1$ to K are fixed at FSN strategies. We denote by $G_k^i(x_k, \psi_k^1, \psi_k^2)$ the cost incurred by player i in this game when players use $(\psi_k^1, \psi_k^2) \in \Psi_k^1 \times \Psi_k^2$ at stage k, then we have

$$\begin{aligned} G_k^i(x_k, \psi_k^1, \psi_k^2) := g_k^i(x_k, (\gamma_k^1(x_k), \gamma_k^2(x_k)), (v_k^1, v_k^2)) + \sum_{\tau=k+1}^{K-1} g_\tau^i(x_\tau, (\gamma_\tau^{1\star}(x_\tau), \gamma_\tau^{2\star}(x_\tau)), (v_\tau^{1\star}, v_\tau^{2\star})) \\ + g_K^i(x_K, (v_K^{1\star}, v_K^{2\star})), \end{aligned} \tag{8}$$

where $x_{k+1} = f_k(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k))$ and $x_{\tau+1} = f_\tau(x_\tau, \gamma_\tau^{1\star}(x_\tau), \gamma_\tau^{2\star}(x_\tau))$ for $\tau = k+1, \cdots, K-1$. Following the state-additive structure of the cost functions, G_k^i can be written recursively backwards for $k = K-1, \cdots, 0$ as

$$G_k^i(x_k, \psi_k^1, \psi_k^2) = g_k^i(x_k, (\gamma_k^1(x_k), \gamma_k^2(x_k)), (v_k^1, v_k^2)) + G_{k+1}^i(f_k(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)), \psi_{k+1}^{1\star}, \psi_{k+1}^{2\star}), \tag{9a}$$

$$G_K^i(x_k, \psi_K^{1\star}, \psi_K^{2\star}) = g_K^i(x_K, v_K^{1\star}, v_K^{2\star}). \tag{9b}$$

Using the separability Assumption 2, we introduce the auxiliary cost of player i as

$$\tilde{G}_{u,k}^i(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)) := g_{u,k}^i(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)) + G_{k+1}^i(f_k(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)), \psi_{k+1}^{1\star}, \psi_{k+1}^{2\star}). \tag{9c}$$

Then, using (9), player i's cost (8) is given by

$$G_k^i(x_k, \psi_k^1, \psi_k^2) = \tilde{G}_{u,k}^i(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)) + g_{v,k}^i(x_k, (v_k^1, v_k^2)). \tag{10}$$

4.1 Simultaneous interaction

The third-stage decision problem at any time instant $k \in \mathsf{K}_l$ given by (5c)–(5d) translates as

$$G_k^1(x_k, (\gamma_k^1(x_k), v_k^{1*}), (\gamma_k^2(x_k), v_k^{2*})) \leq G_k^1(x_k, (\gamma_k^1(x_k), v_k^1), (\gamma_k^2(x_k), v_k^{2*})), \; \forall v_k^1 \in V_k^1(v_k^{2\star}), \tag{11a}$$
$$G_k^2(x_k, (\gamma_k^1(x_k), v_k^{1*}), (\gamma_k^2(x_k), v_k^{2*})) \leq G_k^2(x_k, (\gamma_k^1(x_k), v_k^{1*}), (\gamma_k^2(x_k), v_k^2)), \; \forall v_k^2 \in V_k^2(v_k^{1\star}). \tag{11b}$$

Clearly, from (10), the above third-stage problem results in a static parametric (in x_k) nonzero-sum game denoted by $\Gamma_k(x_k) := \langle \{1,2\}, \mathsf{V}_k(x_k), \{g_{v,k}^i(x_k, v_k^1, v_k^2)\}_{i\in\{1,2\}} \rangle$, where each player $i \in \{1,2\}$ solves

$$\min_{v_k^i \in \mathbb{R}^{s_i}} \; g_{v,k}^i(x_k, (v_k^i, v_k^{-i\star})),$$
$$\text{subject to } h_k^i(x_k, (v_k^i, v_k^{-i\star})) \geq 0, \; v_k^i \geq 0. \tag{12}$$

Introduce the Lagrangian associated with the above constrained optimization problem as

$$L_k^i(x_k, v_k^i, \mu_k^i) := g_{v,k}^i(x_k, (v_k^i, v_k^{-i\star})) - {\mu_k^i}' h_k^i(x_k, (v_k^i, v_k^{-i\star})),$$

where μ_k^i is the Lagrange multiplier associated with the constraint $h_k^i(x_k, (v_k^i, v_k^{-i\star})) \geq 0$. The KKT conditions associated with the third-stage interaction can be represented compactly as the following parametric (in x_k) complementarity problem at time instant k, denoted by $\mathrm{pCP}_k(x_k)$:

$$\mathrm{pCP}_k(x_k): \quad 0 \leq \begin{bmatrix} \nabla L_k(x_k, \mathsf{v}_k^\star, \upmu_k^\star) \\ h_k(x_k, \mathsf{v}_k^\star) \end{bmatrix} \perp \begin{bmatrix} \mathsf{v}_k^\star \\ \upmu_k^\star \end{bmatrix} \geq 0, \tag{13}$$

where

$$\nabla L_k(x_k, \mathsf{v}_k^\star, \upmu_k^\star) = \nabla g_{v,k}(x_k, \mathsf{v}_k^\star) - {\upmu_k^\star}' \nabla h_k(x_k, \mathsf{v}_k^\star), \tag{14a}$$
$$\nabla g_{v,k}(x_k, \mathsf{v}_k^\star) = \begin{bmatrix} \nabla_{v_k^1} g_{v,k}^1(x_k, \mathsf{v}_k^\star) \\ \nabla_{v_k^2} g_{v,k}^2(x_k, \mathsf{v}_k^\star) \end{bmatrix}, \upmu_k^\star = \begin{bmatrix} \mu_k^{1\star} \\ \mu_k^{2\star} \end{bmatrix}, \tag{14b}$$
$$\nabla h_k(x_k, \mathsf{v}_k^\star) = \begin{bmatrix} \nabla_{v_k^1} h_k^1(x_k, \mathsf{v}_k^\star) \\ \nabla_{v_k^2} h_k^2(x_k, \mathsf{v}_k^\star) \end{bmatrix}, h_k(x_k, \mathsf{v}_k^\star) = \begin{bmatrix} h_k^1(x_k, \mathsf{v}_k^\star) \\ h_k^2(x_k, \mathsf{v}_k^\star) \end{bmatrix}. \tag{14c}$$

Remark 5. Recall that in our class of games, the players interact only simultaneously at terminal time K. Therefore, the decision problem at K is similar to (12) and results in the complementarity problem $\mathrm{pCP}_K(x_K)$.

We denote the solution of (13) by $(\mathsf{v}_k^\star, \upmu_k^\star) := \mathtt{SOL}(\mathrm{pCP}_k(x_k))$. It is well-known that a static game can admit multiple Nash equilibria. As a consequence, the upstream decision problems have to be solved for each one of these solutions; see Remark 4. This can be avoided if $\mathrm{pCP}_k(x_k)$ has a unique solution at every time instant $k \in \mathsf{K}$. To ensure this, we have from [34, Theorem 2] the following assumption on the cost functions $g_{v,k}^i$ associated with the third stage game $\Gamma_k(x_k)$.

Assumption 3. For every $(k, x_k) \in \mathsf{K} \times \mathsf{X}_k$ the cost functions $g_{v,k}^i(x_k, .) : \mathbb{R}^s \to \mathbb{R}$, $i \in \{1,2\}$ are *diagonally strictly convex* (DSC). That is, for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^s$ we have

$$(\boldsymbol{\alpha} - \boldsymbol{\beta})' \nabla g_{v,k}(x_k, \boldsymbol{\beta}) + (\boldsymbol{\beta} - \boldsymbol{\alpha})' \nabla g_{v,k}(x_k, \boldsymbol{\alpha}) < 0. \tag{15}$$

Remark 6. A sufficient condition for the cost functions to satisfy the DSC property is that the symmetric matrix $[G(x_k, \mathsf{v}_k) + G'(x_k, \mathsf{v}_k)]$ be positive definite for every $\mathsf{v}_k \in \mathbb{R}^s$, where $G(x_k, \mathsf{v}_k)$ is the Jacobian of $\nabla g_{v,k}(x_k, \mathsf{v}_k)$ with respect to v_k.

Lemma 1. *Consider the CNZDG described by* (1)–(4)*. Let Assumptions 1, 2 and 3 hold. Then, for every* $(k, x_k) \in \mathsf{K} \times \mathsf{X}_k$*,* $\mathtt{SOL}(\mathrm{pCP}_k(x_k))$ *is unique (a singleton), and the FSN control actions* $\mathsf{v}_k^\star$ *are obtained as* $(\mathsf{v}_k^\star, \upmu_k^\star) = \mathtt{SOL}(\mathrm{pCP}_k(x_k))$.

Proof. From Assumption 1.(i), the constraint sets $V_K(x_K)$ are nonempty, convex, closed and bounded for every $i \in \{1, 2\}$ and $x_K \in \mathsf{X}_K$. Further, from Assumption 3, the cost functions are strictly diagonally convex. Then, from [34, Theorem 2], $\Gamma_K(x_K)$ is a convex game, and in particular, the Nash equilibrium obtained as $(\mathsf{v}_K^\star, \mu_K^\star) = \mathtt{SOL}(\mathrm{pCP}_K(x_K))$ is unique for all $x_K \in \mathsf{X}_k$. For any non-terminal instant $k \in \mathsf{K}_l$ and any $x_k \in \mathsf{X}_k$, due to Assumption 2 on separability of cost functions, the optimization problems (11a)–(11b) result in a static parametric nonzero-sum game $\Gamma_k(x_k)$. Again, from Assumption 3 the cost functions $g^i_{v,k}$, $i \in \{1, 2\}$ satisfy the DSC property, which implies that the Nash equilibrium obtained by solving the complementary problem (13) is unique. Therefore, the FSN control actions $\mathsf{v}_k^\star$ are unique for any $x_k \in \mathsf{X}_k$. □

4.2 Sequential interaction

In the second stage, the follower solves the optimization problem (5e) for any leader's announcement $\gamma_k^1 \in \Gamma_k^1$. Again, from Assumption 2 on separability of cost functions, the follower's optimal response is obtained as

$$\begin{aligned} \mathsf{R}_k^2(\gamma_k^1) &= \underset{\gamma_k^2 \in \Gamma_k^2}{\arg\min}\, G_k^2(x_k, (\gamma_k^1(x_k), v_k^{1\star}), (\gamma_k^2(x_k), v_k^{2\star})) \\ &= \underset{\gamma_k^2 \in \Gamma_k^2}{\arg\min}\, \tilde{G}_{u,k}^2(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)). \end{aligned} \tag{16}$$

Next, in the first stage, the leader solves the optimization problem (5f) considering the follower's best response. That is, the leader's optimization problem is given by

$$\gamma_k^{1\star} \in \underset{\gamma_k^1 \in \Gamma_k^1}{\arg\min}\, G_k^1(x_k, (\gamma_k^1(x_k), v_k^{1\star}), ((\mathsf{R}_k^2 \circ \gamma_k^1)(x_k), v_k^{2\star})) \tag{17}$$

Again, from Assumption 2 the leader's FSN solution $\gamma_k^{1\star}$ satisfies

$$\tilde{G}_{u,k}^1(x_k, \gamma_k^{1\star}(x_k), (\mathsf{R}_k^2 \circ \gamma_k^{1\star})(x_k)) \le \tilde{G}_{u,k}^1(x_k, \gamma_k^1(x_k), (\mathsf{R}_k^2 \circ \gamma_k^1)(x_k)),\ \forall \gamma_k^1 \in \Gamma_k^1. \tag{18}$$

Remark 7. In our model, Assumption 2 (separability) and Assumption 3 (DSC property) enable to compute the third stage FSN decisions uniquely as $(\mathsf{v}_k^\star, \mu_k^\star) = \mathtt{SOL}(\mathrm{pCP}_k(x_k))$ at every time instant $k \in \mathsf{K}$. As a result, the third stage decisions taken at time instant k are decoupled from the first and second stage decisions (u_k^1, u_k^2) taken at k. However, they are influenced by the sequential decisions (u_{k-1}^1, u_{k-1}^2) taken at the upstream time instant $k-1$ indirectly through the state variable x_k.

Remark 8. In accordance with the definition of FSN solution, in (17), it is assumed that the optimal response set of the follower is a singleton set. This condition is readily met when $\tilde{G}_{u,k}^2(x_k, \gamma_k^1(x_k), .)$ is a strictly convex function on U_k^2; see also [2, Theorem 7.1 and Theorem 7.2].

Remark 9. When the leader's optimization problem (19b) at any stage k has more than one solution $\gamma_k^{1\star}$, then the leader is indifferent between them as all the solutions give her the same stage wise cost. However, the follower's optimal response $\gamma_k^{2\star}$ may vary with the leader's choice and consequently the next stage state x_{k+1} given by (1). But x_{k+1} determines the decisions $(v_{k+1}^{1\star}, v_{k+1}^{2\star})$ at stage $k+1$, which implies that stage-wise multiplicities in the leader's decisions could result in different global FSN costs for her. More precisely, if $(\hat{\psi}^1, \hat{\psi}^2)$ and $(\bar{\psi}^1, \bar{\psi}^2)$ are two FSN solutions for CNZDG such that $J_1(x_0, \hat{\psi}^1, \hat{\psi}^2) < J_1(x_0, \bar{\psi}^1, \bar{\psi}^2)$, then the leader would definitely prefer $\hat{\psi}^1$ over $\bar{\psi}^1$ and tries to enforce the component $\hat{\gamma}_k^{1\star}$ of the strategy $\hat{\psi}^1$ at each stage k of the game. Note that due to uniqueness of the simultaneous decision variables (under Assumption 3) and uniqueness of follower's response, the leader's cost is unambiguously determined by $\gamma_k^{1\star}$ at every stage. So, this reasoning leads to a total ordering among different FSN solutions of CNZDG from leader's perspective. In this view, we call a FSN solution $(\hat{\psi}^1, \hat{\psi}^2)$ of CNZDG *admissible* (also see [2, Definition 3.30]) if there exists no other FSN solution $(\bar{\psi}^1, \bar{\psi}^2)$ with the property $J_1(x_0, \bar{\psi}^1, \bar{\psi}^2) < J_1(x_0, \hat{\psi}^1, \hat{\psi}^2)$. Clearly, the leader can resolve the multiplicity issue in her optimization problem (19b) by enforcing the admissible ones on the follower at every stage.

4.3 Sufficient conditions for a FSN solution

In this subsection, we provide sufficient conditions for the existence of a FSN solution. The next theorem utilizes the reachable sets $\mathsf{X}_k,\ k \in \mathsf{K}$ to transform conditions (5a)–(5f) in Definition 3 into a recursive characterization of FSN solution.

Theorem 1. *Consider the CNZDG described by (1)–(4). Let Assumptions 1, 2 and 3 hold true. If there exist functions $W^i(k,.) : \mathsf{X}_k \to \mathbb{R},\ k \in \mathsf{K}\ i \in \{1,2\}$ such that for all $k \in \mathsf{K}_l$ and $\gamma_k^1 \in \Gamma_k^1,\ g_{u,k}^2(x_k, \gamma_k^1(x_k), \cdot) + W^2(k+1, f_k(x_k, \gamma_k^1(x_k), \cdot))$ is strictly convex function on U_k^2 and for all $k \in \mathsf{K},\ W^i(k,\cdot)$ satisfy the following (backward) recursive relations:*

1. *At period K*

$$W^i(K, x_K) = g_K^i(x_K, (v_K^{1\star}, v_K^{2\star})),\ i \in \{1,2\}, \tag{19a}$$

where $(\mathsf{v}_K^\star, \mu_K^\star) = \mathit{SOL}(\mathrm{pCP}(x_K))$.

2. *At periods $k = K-1, \cdots, 1, 0$*

$$\begin{aligned} W^i(k, x_k) = &\ g_k^i\big(x_k, \gamma_k^{1*}(x_k), \gamma_k^{2*}(x_k), (v_k^{1\star}, v_k^{2\star})\big) \\ &+ W^i\big(k+1, f_k(x_k, \gamma_k^{1*}(x_k), \gamma_k^{2*}(x_k))\big),\ i \in \{1,2\}, \end{aligned} \tag{19b}$$

where

$$(\mathsf{v}_k^\star, \mu_k^\star) = \mathit{SOL}(\mathrm{pCP}_k(x_k)), \tag{19c}$$

$$\mathsf{R}_k^2(\gamma_k^1) = \underset{\gamma_k^2 \in \Gamma_k^2}{\mathrm{argmin}} \Big\{ g_{u,k}^2\big(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k)\big) + W^2\big(k+1, f_k(x_k, \gamma_k^1(x_k), \gamma_k^2(x_k))\big) \Big\}, \tag{19d}$$

$$\gamma_k^{1\star} \in \underset{\gamma_k^1 \in \Gamma_k^1}{\mathrm{argmin}} \Big\{ g_{u,k}^1\big(x_k, \gamma_k^1(x_k), (\mathsf{R}_k^2 \circ \gamma_k^1)(x_k)\big) + W^1\big(k+1, f_k(x_k, \gamma_k^1(x_k), (\mathsf{R}_k^2 \circ \gamma_k^1)(x_k))\big) \Big\}, \tag{19e}$$

$$\gamma_k^{2\star} = \mathsf{R}_k^2(\gamma_k^{1\star}). \tag{19f}$$

Then, the pair of strategies $\{\psi^{i\star} = ((\gamma_k^{i\star}(x_k), v_k^{i\star})_{K_l}, v_K^{i\star}), i = 1,2\}$ constitutes a FSN solution for CNZDG. Further, every such FSN solution is strongly time consistent.

Proof. We prove the theorem by backward induction. First, we start with the set of inequalities (5a)–(5b) defined at terminal time K. Since they have to hold true for all admissible actions $\{\psi_{[0,K-1]}^i,\ i \in \{1,2\}\}$, it must be that they hold true for all the states x_K that are reachable by the utilization of a combination of these admissible actions. Then, conditions (5a)–(5b) are equivalent to solving a static game where each player $i \in \{1,2\}$ solves the following problem:

$$g_K^i(x_K, (v_K^{i\star}, v_K^{-i\star})) \leq g_K^i(x_K, (v_K^i, v_K^{-i\star}),\ \forall v_K^i \in V_K^i(v_k^{-i\star}). \tag{20}$$

From Assumption 3 and Lemma 1 the solution of (20), given by $(\mathsf{v}_K^\star, \mu_K^\star) = \mathtt{SOL}(\mathrm{pCP}_K(x_K))$, is unique and provides the FSN solution of the subgame at terminal time K. Further, we notice that the FSN decisions $(v_K^{1\star}, v_K^{2\star})$ depend only on the current state x_K and not on the past values of the state variable (including x_0).

Next, we assume that the theorem holds up to time period $k+1 \in \mathsf{K}_r$ backwards in time, that is, the strategies $\{\psi_{[k+1,K]}^{i\star},\ i \in \{1,2\}\}$ obtained from (19b)–(19f), satisfy the Definition 3 of FSN solution. The FSN cost accumulated by player $i \in \{1,2\}$ for any $x_{k+1} \in \mathsf{X}_{k+1}$ is given by

$$W^i(k+1, x_{k+1}) = \sum_{\tau=k+1}^{K} g_\tau^i(x_\tau, \psi_\tau^{1\star}, \psi_\tau^{2\star}).$$

We show that for period k the theorem holds true. The set of inequalities (5c)–(5f) defined at time k, by fixing the downstream decisions at the FSN solutions $\{\psi^{i\star}_{[k+1,K]},\ i \in \{1,2\}\}$, correspond to the quasi-hierarchical interaction involving three decision stages. These inequalities must hold true for all admissible actions $\{\psi^i_{[0,k-1]},\ i \in \{1,2\}\}$, and as a result, they hold true for all the states $x_k \in \mathsf{X}_k$ that are reachable by the utilization of a combination of these admissible actions. Between the time instants k and $k+1$, players interact in three decision stages where each player $i \in \{1,2\}$ seeks to minimize the cost functional

$$g^i_k(x_k,(u^1_k,u^2_k),(v^1_k,v^2_k)) + W^i(k+1,x_{k+1}),$$

subject to the constraints $h^i_k(x_k,\mathsf{v}_k) \geq 0$, $v^i_k \geq 0$ and with $x_{k+1} = f_k(x_k,u^1_k,u^2_k)$. From the separability Assumption 2 and Lemma 1, the unique Nash outcome $\mathsf{v}^\star_k$ of the third stage problem is given by (19c). In the second stage, under the assumption of strict convexity of $g^2_{u,k}(x_k,\gamma^1_k(x_k),\cdot\,) + W^2(k+1, f_k(x_k,\gamma^1_k(x_k),.)$ on U^2_k, the optimal response R^2_k defined by (19d) is unique. In the first stage, using the follower's response, the leader's problem is given by (19e). Then, $W^i(k,x_k)$, given by (19b), denotes precisely the FSN cost of player i in the subgame starting at (k,x_k). We note that the FSN decisions at each time instant k depend only on the current x_k and not on the past values of the state variable (including x_0). The strong time consistency property of the FSN solution is a direct consequence of the backward recursive nature of construction of the solution. □

Remark 10. We note that Theorem 1 provides only sufficient conditions for the existence of FSN solution for CNZDG. This is because the strict convexity of $g^2_{u,k}(x_k,\gamma^1_k(x_k),\cdot\,) + W^2(k+1, f_k(x_k,\gamma^1_k(x_k),\cdot))$ over U^2_k and DSC Assumption 3 on cost functions $g^i_{v,k}$ are sufficient for unique follower's response and unique Nash equilibrium for the static game $\Gamma_k(x_k)$, respectively, at every time period.

4.4 Parametric feedback Stackelberg solution and its relation to FSN solution

In this subsection, we show that a FSN solution is closely related to a feedback Stackelberg solution of an unconstrained parametric nonzero-sum difference game (pNZDG) involving only sequential interactions. To this end, we first define a pNZDG associated with the CNZDG, with parameters $\{\mathsf{w}_\tau := (w^1_\tau, w^2_\tau) \in \mathbb{R}^{s_1+s_2}, \theta_\tau := (\theta^1_\tau,\theta^2_\tau) \in \mathbb{R}^{c_1+c_2}, \tau \in \mathsf{K}\}$ as follows:

$$\begin{aligned}\text{pNZDG}: \min_{\tilde{\mathsf{u}}^i} \Big\{ & \bar{J}_i\big(x_0,\tilde{\mathsf{u}}^1,\tilde{\mathsf{u}}^2;\{(\mathsf{w}_\tau,\theta_\tau)\}_{\tau\in\mathsf{K}}\big) \\ &= g^i_K(x_K,(w^1_K,w^2_K)) - \theta^{i'}_K h^i_K(x_K,\mathsf{w}_K) \\ &\quad + \sum_{k=0}^{K-1}\big(g^i_k((x_k,(u^1_k,u^2_k)),(w^1_k,w^2_k)) - \theta^{i'}_k h^i_k(x_k,\mathsf{w}_k)\big)\Big\}, \qquad (21a)\\ & \text{subject to } x_{k+1} = f_k(x_k,u^1_k,u^2_k),\ k \in \mathsf{K}_l,\ x_0 \text{ given.} \qquad (21b)\end{aligned}$$

In pNZDG, the players interact sequentially in the decision variables (u^1_k,u^2_k) at every time instant $k \in \mathsf{K}_l$. Under feedback information structure, the control action u^i_k of player i at instant k is denoted by $u^i_k := \xi^i_k(x_k) \in \mathsf{U}^i_k$, where $\xi^i_k : \mathbb{R}^n \to \mathsf{U}^i_k$ is a measurable mapping with the set of all such mappings be denoted by Ξ^i_k. The feedback strategy profile of player i be denoted by ξ^i and the corresponding strategy set by $\Xi^i = \prod_{k=0}^{K-1}\Xi^i_k$. As there are no constraints, the reachable set $\bar{\mathsf{X}}_k$ is entire $\mathbb{R}^n$. The definition of parametric feedback Stackelberg solution for pNZDG follows from the standard feedback Stackelberg solution [2], which is given as follows.

Definition 4 ([2, Definition 3.29]). A pair $(\xi^{1\star},\xi^{2\star}) \in (\Xi^1,\Xi^2)$ constitutes a *parametric feedback Stackelberg* (pFS) solution for pNZDG if the following conditions are satisfied

1. For all $\xi^i_{[0,k-1]} \in \Xi^i_{[0,k-1]}$, $i \in \{1,2\}$ with $k = K-1,\cdots,1$

$$\bar{J}_1\big(x_0,(\xi^1_{[0,k-1]},\xi^{1\star}_k,\xi^{1\star}_{[k+1,K-1]}),(\xi^2_{[0,k-1]},\xi^{2\star}_k,\xi^{2\star}_{[k+1,K-1]});\{(\mathsf{w}_\tau,\theta_\tau)\}_{\tau\in\mathsf{K}}\big)$$

$$= \min_{\xi_k^1 \in \Xi_k^1} \max_{\hat{\xi}_k^2 \in \bar{\mathsf{R}}_k^2(\xi_k^1)} \bar{J}_1\big(x_0, (\xi^1_{[0,k-1]}, \xi^1_k, \xi^{1\star}_{[k+1,K-1]}), (\xi^2_{[0,k-1]}, \hat{\xi}^2_k, \xi^{2\star}_{[k+1,K-1]}); \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau\in\mathsf{K}}\big), \tag{22a}$$

where $\bar{\mathsf{R}}_k^2(\xi_k^1)$ is the optimal response set of the follower at stage k, defined by

$$\bar{\mathsf{R}}_k^2(\xi_k^1) := \arg\min_{\xi_k^2 \in \Xi_k^2} \bar{J}_2\big(x_0, (\xi^1_{[0,k-1]}, \xi^1_k, \xi^{1\star}_{[k+1,K-1]}), (\xi^2_{[0,k-1]}, \xi^2_k, \xi^{2\star}_{[k+1,K-1]}); \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau\in\mathsf{K}}\big), \tag{22b}$$

2. The optimal response set $\bar{\mathsf{R}}_k^2(\xi_k^{1\star})$ is a singleton set.

Further, $J_1(x_0, \xi^{1\star}, \xi^{2\star}; \{(\mathsf{w}_\tau \theta_\tau)\}_{\tau\in\mathsf{K}})$ and $J_2(x_0, \xi^{1\star}, \xi^{2\star}; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau\in\mathsf{K}})$ represent the parametric feedback Stackelberg costs incurred by the leader and the follower, respectively, with parameters $\{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau\in\mathsf{K}}$.

Remark 11. Following Assumption 2, the cost functions g_k, $k \in \mathsf{K}_l$ in (21a) have the separable structure, that is, they do not contain cross-terms involving the decision variables (u_k^1, u_k^2) and the parameters (w_k^1, w_k^2).

The recursive formulation of a pFS solution for pNZDG follows from (22a) and (22b). Using the standard feedback-Stackelberg solution [2, Theorem 7.2], the next lemma reveals the structure of a pFS solution for pNZDG. We omit the proof of the lemma as it follows directly from [2, Theorem 7.2].

Lemma 2. *Consider the pNZDG described by* (21)*. Let Assumption 2 hold. If there exist functions* $W_p^i(k,.) : \bar{\mathsf{X}}_k \to \mathbb{R}$, $k \in \mathsf{K}$, $i \in \{1,2\}$ *such that for all* $k \in \mathsf{K}_l$ *and* $\forall \xi_k^1 \in \Xi_k^1$, $g_{u,k}^2(x_k, \xi_k^1(x_k), \cdot) + W_p^2(k+1, f_k(x_k, \xi_k^1(x_k), \cdot))$ *is strictly convex on* U_k^2 $(k \in \mathsf{K}_l)$ *and* $\forall k \in \mathsf{K}$, $W_p^i(k, \cdot)$ *satisfy the following (backward) recursive relations:*

1. *At period* K

$$W_p^i(K, x_K; (\mathsf{w}_K, \theta_K)) = g_K^i(x_K, \mathsf{w}_K) - {\theta_K^i}' h_K^i(x_K, \mathsf{w}_K),\ i \in \{1,2\}. \tag{23a}$$

2. *At periods* $k = K-1, \cdots, 1, 0$

$$\begin{aligned} W_p^i\big(k, x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k}^K\big) &= g_k^i(x_k, \xi_k^{1\star}(x_k), \xi_k^{2\star}(x_k), \mathsf{w}_k) \\ &- {\theta_k^i}' h_k^i(x_k, \mathsf{w}_k) + W_p^i\big(k+1, f_k(x_k, \xi_k^{1\star}(x_k), \xi_k^{2\star}(x_k)); \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K\big), \end{aligned} \tag{23b}$$

where

$$\begin{aligned} \bar{\mathsf{R}}_k^2(\xi_k^1) = \underset{\xi_k^2 \in \Xi_k^2}{\operatorname{argmin}} \Big\{ & g_{u,k}^2(x_k, \xi_k^1(x_k), \xi_k^2(x_k)) \\ & + W_p^2\big(k+1, f_k(x_k, \xi_k^1(x_k), \xi_k^2(x_k)); \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K\big)\Big\}, \end{aligned} \tag{23c}$$

$$\begin{aligned} \xi_k^{1\star} \in \underset{\xi_k^1 \in \Xi_k^1}{\operatorname{argmin}} \Big\{ & g_{u,k}^1\big(x_k, \xi_k^1(x_k), (\bar{\mathsf{R}}_k^2 \circ \xi_k^1)(x_k)\big) \\ & + W_p^1\big(k+1, f_k\big(x_k, \xi_k^1(x_k), (\bar{\mathsf{R}}_k^2 \circ \xi_k^1)(x_k)\big); \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K\big)\Big\}, \end{aligned} \tag{23d}$$

$$\xi_k^{2\star} = \bar{\mathsf{R}}_k^2(\xi_k^{1\star}). \tag{23e}$$

Then, the pair of strategies $\{\xi^{1\star},\ \xi^{2\star}\}$ *constitutes a pFS solution for pNZDG. Further, every such solution is strongly time consistent, and admits a parametric representation given by*

$$\big\{\xi_k^{i\star}(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K),\ k \in \mathsf{K}_l,\ i \in \{1,2\}\big\}. \tag{24}$$

Remark 12. We notice that for any stage $k \in \mathsf{K}_l$, and any $x_k \in \bar{\mathsf{X}}_k$, $W_p^i(k, x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k}^K)$ denotes the cost incurred by player i using the pFS strategies $(\xi^{1\star}_{[k,K-1]}, \xi^{2\star}_{[k,K-1]})$ and thus it represents the value function for player i. Here, the value functions in (23b) are obtained by solving the game recursively backwards, and depend on the past values of the parameters. For this reason, we represent the value function at time instant k explicitly in parametric form as $W_p^i(k, x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k}^K)$.

Next, we present the main result of this section where we show that a FSN solution for CNZDG can be obtained from a pFS solution for pNZDG through a specific choice of the parameters.

Theorem 2. *Consider the CNZDG described by (1)–(4) and the pNZDG described by (21). Let Assumptions 1, 2 and 3 hold true. Assume there exist functions* $W^i(k,.) : \mathsf{X}_k \to \mathbb{R}$ *and* $W_p^i(k,.) : \bar{\mathsf{X}}_k \to \mathbb{R}$ *for* $k \in \mathsf{K}$, $i \in \{1,2\}$ *satisfying conditions (19) and (23), respectively. Further, assume that for all* $k \in \mathsf{K}_l$ *the functions* $g^2_{u,k}(x_k, \gamma^1_k(x_k), \cdot) + W^2(k+1, f_k(x_k, \gamma^1_k(x_k), \cdot))$, $\forall \gamma^1_k \in \Gamma^1_k$ *and* $g^2_{u,k}(x_k, (\xi^1_k(x_k), \cdot), \cdot) + W^2_p(k+1, f_k(x_k, \xi^1_k(x_k), \cdot))$, $\forall \xi^1_k \in \Xi^1_k$ *are strictly convex on* U^2_k. *Let for player* $i \in \{1,2\}$, $\{\xi^{i\star}_k(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}^K_{\tau=k}), k \in \mathsf{K}_l$ *denote a pFS solution for PNZDG. Then, a FSN solution for player* $i \in \{1,2\}$ *satisfies*

$$\gamma^{i\star}_k(x_k) = \xi^{i\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}), \ \forall k \in \mathsf{K}_l, \tag{25a}$$

where $(\mathsf{v}^\star_k, \mu^\star_k) = \mathtt{SOL}(\mathrm{pCP}_k(x_k))$ *for all* $k \in \mathsf{K}$, *and* $\{x_k,\ k \in \mathsf{K}\}$ *is the state trajectory generated by the difference equation*

$$x_{k+1} = f_k\big(x_k, \xi^{1\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}), \xi^{2\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1})\big).$$

Further, for all $k \in \mathsf{K}$ *and* $i \in \{1,2\}$ *the (value) function* $W^i(k,.) : \mathsf{X}_k \to \mathbb{R}$ *satisfies*

$$W^i(k, x_k) = W^i_p(k, x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k}), \quad x_k \in \mathsf{X}_k. \tag{25b}$$

Proof. We prove the theorem using (backward) induction principle. The terminal game associated with CNZDG at time instant K, with $x_K \in \mathsf{X}_K$, is characterized by (19a). Following Assumption 3, and Lemma 1, there exists a unique $(\mathsf{v}^\star_K, \mu^{i\star}_K) = \mathtt{SOL}(\mathrm{pCP}_K(x_K))$. Upon setting the parameters (w_K, θ_K) equal to $(\mathsf{v}^\star_K, \mu^\star_K)$ in (23a), and from the FSN costs of players (19a) along with the complimentarity condition $\mu^{i\star\prime}_K h^i_K(x_K, \mathsf{v}^\star_K) \equiv 0$ for $i \in \{1,2\}$, we get

$$W^i(K, x_K) = g^i_K(x_K, (v^{1\star}_K, v^{2\star}_K)) - \mu^{i\star\prime}_K h^i_K(x_K, \mathsf{v}^\star_K) = W^i_p(K, x_K; (\mathsf{v}^\star_K, \mu^\star_K)).$$

Therefore, the statement of the theorem at terminal time is verified. Next, assuming that the theorem holds for time instant $k+1$ we show it also holds for instant k, with $k = K-1, \cdots, 0$. Since the theorem holds for time $k+1$, we have for time instants $l = k+1$ to $l = K-1$, FSN solution of player $i \in \{1,2\}$ is related to the pFS solution as

$$\gamma^{i\star}_l(x_l) = \xi^{i\star}_l(x_l; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=l+1}), \tag{26}$$

and the FSN costs of the player i in the subgame starting at $(k+1, x_{k+1})$ is given by

$$W^i(k+1, x_{k+1}) = W^i_p(k+1, x_{k+1}; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}), \tag{27}$$

where $(\mathsf{v}^\star_\tau, \mu^\star_\tau) = \mathtt{SOL}(\mathrm{pCP}_\tau(x_\tau))$ for $\tau = k+1, \cdots, K$. Now, we consider the static game characterized by conditions (5c)–(5f) at time instant k, for any $x_k \in \mathsf{X}_k$, with player i cost function given by

$$g^i_k(x_k, (u^1_k, u^2_k), (v^1_k, v^2_k)) + W^i(k+1, f_k(x_k, u^1_k, u^2_k)). \tag{28}$$

From Assumption 3, and Lemma 1, the third stage decisions are obtained as the unique Nash outcome given by $(\mathsf{v}^\star_k, \mu^\star_k) = \mathtt{SOL}(\mathrm{pCP}_k(x_k))$. Using (27) in the second and first stage decision problems (19d) and (19e), respectively, we get

$$\begin{aligned} \mathsf{R}^2_k(\gamma^1_k) = \operatorname*{argmin}_{\gamma^2_k \in \Gamma^2_k} \Big\{ & g^2_{u,k}\big(x_k, \gamma^1_k(x_k), \gamma^2_k(x_k)\big) \\ & + W^2_p\big(k+1, f_k(x_k, \gamma^1_k(x_k), \gamma^2_k(x_k)); \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}\big)\Big\}, \end{aligned} \tag{29}$$

$$\begin{aligned} \gamma^{1\star}_k \in \operatorname*{argmin}_{\gamma^1_k \in \Gamma^1_k} \Big\{ & g^1_{u,k}\big(x_k, \gamma^1_k(x_k), (\mathsf{R}^2_k \circ \gamma^1_k)(x_k)\big) \\ & + W^1_p\big(k+1, f_k(x_k, \gamma^1_k(x_k), (\mathsf{R}^2_k \circ \gamma^1_k)(x_k)); \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}\big)\Big\}. \end{aligned} \tag{30}$$

Now, consider the subgame starting at (k, x_k) for pNZDG with the parameters fixed as $(\mathsf{w}_\tau, \theta_\tau) = (\mathsf{v}^\star_\tau, \mu^\star_\tau) = \mathtt{SOL}(\mathrm{pCP}_\tau(x_\tau))$ for $\tau = k+1, \cdots, K$. The follower's rational response (23c), and the leader's optimization problem (23d) are given by

$$\begin{aligned}
\bar{\mathsf{R}}^2_k(\xi^1_k) = \underset{\xi^2_k \in \Xi^2_k}{\operatorname{argmin}} &\Big\{ g^2_{u,k}\big(x_k, \xi^1_k(x_k), \xi^2_n(x_k)\big) \\
&+ W^2_p\big(k+1, f_k(x_k, \xi^1_n(x_k), \xi^2_n(x_k)); \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}\big) \Big\},
\end{aligned} \tag{31}$$

$$\begin{aligned}
\xi^{1\star}_k \in \underset{\xi^1_k \in \Xi^1_k}{\operatorname{argmin}} &\Big\{ g^1_{u,k}\big(x_k, \xi^1_k(x_k), (\bar{\mathsf{R}}^2_k \circ \xi^1_k)(x_k)\big) \\
&+ W^1_p\big(k+1, f_k\big(x_k, \xi^1_k(x_k), (\bar{\mathsf{R}}^2_k \circ \xi^1_k)(x_k)\big); \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}\big) \Big\}.
\end{aligned} \tag{32}$$

In the follower's optimization problems (29) and (31), from the strict convexity of the objective function over U^2_k we have that the optimal reaction sets $\mathsf{R}^2_k(\gamma^1_k)$ and $\bar{\mathsf{R}}^2_k(\xi^1_k)$ are singleton sets. Further, in these problems, the objective functions have identical functional forms, which implies $\mathsf{R}^2_k(\gamma^1_k) = \bar{\mathsf{R}}^2_k(\gamma^1_k)$ for all $\gamma^1_k \in \Gamma^1_k$. In other words, for all leader's announcements in the strategy space Γ^1_k, the follower's response is identical in both problems. We note that in the leader's optimization problems (30) and (32) the objective functions have identical forms. Consequently, when the leader's strategy space in (30) and (32) is restricted to $\Gamma^1_k \subset \Xi^1_k$, then the leader's optimal strategy set is identical in both problems.

For pNZDG, from Lemma 2, the pFS solution of the optimization problems (31) and (32) is of the form $\xi^{i\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}) \in \Xi^i_k$ for $i = 1, 2$. We recall that the downstream parameters in these strategies are set as $(\mathsf{w}_\tau, \theta_\tau) = (\mathsf{v}^\star_\tau, \mu^\star_\tau) = \mathtt{SOL}(\mathrm{pCP}_\tau(x_\tau))$ for $\tau = k+1, \cdots, K$. At time instant k, using the pFS solutions we get $x_{k+1} = f(x_k, \xi^{1\star}_k(x_k), \xi^{2\star}_k(x_k))$, which is required to satisfy $\mathtt{SOL}(\mathrm{pCP}_\tau(x_{k+1})) \neq \emptyset$ as the downstream parameters are fixed. From Assumption 1.(i), this implies that the pFS strategies at time instant k are also admissible, that is, $\xi^{i\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}) \in \Gamma^i_k$, $i = 1, 2$. Next, following the uniqueness of the follower's response in problems (29) and (31), and coincidence of leader's optimal announcement sets over the strategy set Γ^1_k in problems (30) and (32), at time instant k, the FSN solution is obtained from the pFS solution as

$$\gamma^{i\star}_k(x_k) = \xi^{i\star}_k(x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}),\ i = 1, 2. \tag{33}$$

The value function of player i in the subgame starting at (k, x_k) for CNZDG is given by (19b). From (27) and including the complementarity condition $\mu^{i\star'}_k h^i_k(x_k, \mathsf{v}^\star_k) = 0$ we get

$$\begin{aligned}
W^i(k, x_k) &= g^i_k\big(x_k, \gamma^{1*}_k(x_k), \gamma^{2*}_k(x_k), (v^{1\star}_k, v^{2\star}_k)\big) - \mu^{i\star'}_k h^i_k(x_k, \mathsf{v}^\star_k) + W^i_p\big(k+1, x_{k+1}; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k+1}\big), \\
&= W^i_p\big(k, x_k; \{(\mathsf{v}^\star_\tau, \mu^\star_\tau)\}^K_{\tau=k}\big).
\end{aligned}$$

The last equality follows from (23b) by setting $(\mathsf{w}_k, \theta_k) = (\mathsf{v}^\star_k, \mu^\star_k) = \mathtt{SOL}(\mathrm{pCP}_k(x_k))$. So, the statement of the theorem follows by backward induction. □

Theorem 2 is constructive and provides a relationship between pFS solution for pNZDG and FSN solution for CNZDG. Next, using this relation, we characterize the FSN solution set. Substitution for pFS strategies in (21b) generates the pFS state trajectory and is given by

$$x_{k+1} = f_k\big(x_k, \xi^{1\star}_k(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}^K_{\tau=k+1}), \xi^{2\star}_k(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}^K_{\tau=k+1})\big),\ k \in \mathsf{K}_l. \tag{34}$$

Clearly, for a given initial condition $x_0 \in \mathbb{R}^n$, this trajectory is completely determined by the choice of the parameters $(\mathsf{w}_\tau, \theta_\tau)$, $\tau \in \mathsf{K}$. Let $x_{\mathsf{K}} := \mathrm{col}(x_k)^K_{k=1}$, $\mathsf{w}_{\mathsf{K}} := \mathrm{col}(\mathsf{w}_k)^K_{k=1}$ and $\theta_{\mathsf{K}} := \mathrm{col}(\theta_k)^K_{k=1}$, then the pFS state trajectory is given the following mapping

$$x_{\mathsf{K}} = \mathcal{X}_{\mathrm{pFS}}((\mathsf{w}_{\mathsf{K}}, \theta_{\mathsf{K}}); x_0), \tag{35}$$

where $\mathcal{X}_{\text{pFS}} : \mathbb{R}_+^{(s+c)K} \to \mathbb{R}^{nK}$ is a vector representation of (34) obtained by eliminating x_k in the right hand side. Next, using (13)–(14), we define $\nabla L_{\mathsf{K}}(x_{\mathsf{K}}, \mathsf{w}_{\mathsf{K}}, \theta_{\mathsf{K}}) := \text{col}(\nabla L_k(x_k, \mathsf{w}_k, \theta_k))_{k=1}^K$ and $h_{\mathsf{K}}(x_{\mathsf{K}}, \mathsf{w}_{\mathsf{K}}) := \text{col}(h_k(x_k, \mathsf{w}_k))_{k=1}^K$, and define the following complementarity problem

$$\text{pCP}(x_{\mathsf{K}}) : 0 \leq \begin{bmatrix} \nabla L_{\mathsf{K}}(x_{\mathsf{K}}, \mathsf{w}_{\mathsf{K}}, \theta_{\mathsf{K}}) \\ h_{\mathsf{K}}(x_{\mathsf{K}}, \mathsf{w}_{\mathsf{K}}) \end{bmatrix} \perp \begin{bmatrix} \mathsf{w}_{\mathsf{K}} \\ \theta_{\mathsf{K}} \end{bmatrix} \geq 0. \tag{36}$$

From Assumption 3, and Lemma 1, if the above problem has a solution, then it must be unique. So, $\texttt{SOL}(\text{pCP}) : \mathbb{R}^{nK} \to \mathbb{R}_+^{(c+s)K}$ is a piecewise single-valued mapping. Now, consider the composite map

$$\texttt{SOL}(\text{pCP}) \circ \mathcal{X}_{\text{pFS}} : \mathbb{R}_+^{(s+c)K} \to \mathbb{R}_+^{(s+c)K}. \tag{37}$$

The fixed points of this above map are given by

$$\begin{aligned} \mathsf{Q} := \Big\{ (\mathsf{w}_{\mathsf{K}}, \theta_{\mathsf{K}}) \in \mathbb{R}_+^{(s+c)K} \;\Big|\; & (\mathsf{w}_k, \theta_k) = \texttt{SOL}(\text{pCP}_k(x_k) \neq \emptyset, \\ & k \in \mathsf{K}_r,\ x_0 \in \mathbb{R}^n (\text{given}), \\ & x_{k+1} = f_k\big(x_k, \xi_k^{1\star}(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K), \xi_k^{2\star}(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K)\big),\ k \in \mathsf{K}_l \Big\}. \end{aligned} \tag{38}$$

Notice that the set Q defined in (38) is exactly the set of parameters for which the relations (25) in Theorem 2 holds. The above observation is summarized in the following proposition.

Proposition 1. *Consider the CNZDG described by (1)–(4) and the pNZDG described by (21). Let the assumptions stated in Theorem 2 hold. Assume* $\texttt{SOL}(\text{pCP}_0(x_0)) \neq \emptyset$ *and* $\mathsf{Q} \neq \emptyset$, *then the FSN strategies of the players are given by* $\{\psi^{i\star} \equiv (\{(\gamma_k^{i\star}(x_k), v_k^{i\star})\}_{k \in \mathsf{K}_l\}}, v_K^{i\star}),\ i \in \mathsf{N}\}$ *where the simultaneous decisions* $(v_k^{1\star}, v_k^{2\star})$ *satisfy* $(\mathsf{v}_0^\star, \mu_0^\star) \in \texttt{SOL}(\text{pCP}_0(x_0))$ *and* $(\mathsf{v}_K^\star, \mu_K^\star) \in \mathsf{Q}$, *and the sequential decisions* $(\gamma_k^{1\star}, \gamma_k^{2\star})$ *satisfy* $\gamma_k^{i\star}(x_k) = \xi_k^{i\star}(x_k; \{(\mathsf{v}_\tau^\star, \mu_\tau^\star)\}_{\tau=k+1}^K),\ k \in \mathsf{K}_l$.

Proof. The proof follows from the proof of Theorem 2 and the parameter set (38). □

Remark 13. From Proposition 1, for a given initial condition, the FSN solutions are characterized as the intersection points of two parametric maps (35) and (36) (also see Figure 2). This implies that there may exist none (when $\mathsf{Q} = \emptyset$), a unique solution, multiple solutions, or even a continuum of solutions (when $\mathsf{Q} \neq \emptyset$). In case of multiple solutions, along each FSN state trajectory, we will have unique simultaneous equilibrium solution (due to Assumption 3) at every stage. As the follower's response is also unique for each leader's sequential decision $\gamma_k^{1\star}$, at every stage, the leader can enforce her FSN sequential decision locally at every stage of the game. Further, following Remark 9, leader can order the FSN solutions in terms of her global cost which result in admissible FSN solutions.

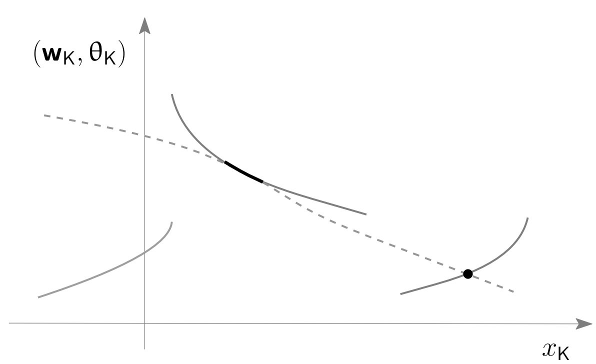

Figure 2: Illustration of set Q defined in (38) as the intersection of the continuous map (35) (gray dashed curve) and piecewise single valued map (36) (gray normal curve).

Remark 14. Following Theorem 2 and Proposition 1 the FSN solution is obtained using the parameter set Q and the pFS solution of the associated unconstrained pNZDG. In particular, the FSN strategies and value functions of the players are obtained as (25a) and (25b) respectively.

Although characterizations of the set Q and the value functions $W_p^i(k,\cdot)$ are available through equations (38) and (23), respectively, obtaining their exact functional forms for general CNZDGs remains a challenge. In the next section, we demonstrate that for the specific class of linear-quadratic games, solving a linear complementary problem allows us to obtain the elements of set Q. Further, the parametric value functions $W_p^i(k,\cdot)$ can be shown to be quadratic functions of the state, enabling the determination of FSN solutions.

5 Linear-quadratic case

In this section, we specialize the results of the previous section to a linear-quadratic setting and provide a method for computing the FSN solution. To this end, we consider a discrete-time finite-horizon linear-quadratic difference game with mixed affine coupled inequality constraints (CLQDG). The state variable evolves according the following discrete-time linear dynamics:

$$x_{k+1} = A_k x_k + B_k^1 u_k^1 + B_k^2 u_k^2, \quad k \in \mathsf{K}_l, \tag{39a}$$

with a given initial state $x_0 \in \mathbb{R}^n$, where $A_k \in \mathbb{R}^{n\times n}$, $B_k^i \in \mathbb{R}^{n\times m_i}$, $u_k^i \in \mathsf{U}_k^i$ for $i \in \{1,2\}$ and $\mathsf{U}_k^i \subset \mathbb{R}^{m_i}$ represents the admissible action set of player i. The mixed coupled constraints (2) of player $i \in \{1,2\}$ for this case, is given by

$$M_k^i x_k + N_k^i \mathsf{v}_k + r_k^i \geq 0,\ v_k^i \geq 0,\ k \in \mathsf{K}, \tag{39b}$$

where $M_k^i \in \mathbb{R}^{c_i\times n}$, $N_k^i \in \mathbb{R}^{c_i\times s}$ and $r_k^i \in \mathbb{R}^{c_i}$. The terminal and instantaneous costs of player $i \in \{1,2\}$ are given by

$$g_K^i(x_K,\mathsf{v}_K) = \tfrac{1}{2}x_K' Q_K^i x_K + {p_K^i}' x_K + \tfrac{1}{2}\mathsf{v}_K' D_K^i \mathsf{v}_K + x_K' L_K^i \mathsf{v}_K + {d_K^i}' \mathsf{v}_K, \tag{39c}$$

$$g_k^i(x_k,\mathsf{u}_k,\mathsf{v}_k) = \tfrac{1}{2}x_k' Q_k^i x_k + {p_k^i}' x_k + \tfrac{1}{2}\sum_{j=1}^{2} {u_k^j}' R_k^{ij} u_k^j + \tfrac{1}{2}\mathsf{v}_k' D_k^i \mathsf{v}_k + x_k' L_k^i \mathsf{v}_k + {d_k^i}' \mathsf{v}_k, \tag{39d}$$

where $R_k^{ij} \in \mathbb{R}^{m_i\times m_j}$ for $k \in \mathsf{K}_l$, and $Q_k^i \in \mathbb{R}^{n\times n}$, $Q_k^i = Q_k^{i'}$, $p_k^i \in \mathbb{R}^n$, $D_k^i \in \mathbb{R}^{s_i\times s_i}$, $L_k^i \in \mathbb{R}^{n\times s_i}$, $d_k^i \in \mathbb{R}^{s_i}$ for $k \in \mathsf{K}$. Similar to Assumption 1, we have the following assumptions for CLQDG.

Assumption 4.

(i) the admissible action sets $\mathsf{U}_k^i \subset \mathbb{R}^{m_i}$ for $k \in \mathsf{K}_l$, $i \in \{1,2\}$ are such that the feasible action sets $\mathsf{V}(x_k) := \big\{\mathsf{v}_k \in \mathbb{R}^s \mid M_k^i x_k + N_k^i \mathsf{v}_k + r_k^i \geq 0,\ i = 1,2, \mathsf{v}_k \geq 0\big\}$ for all $k \in \mathsf{K}$, $i \in \{1,2\}$ are nonempty and bounded.

(ii) The matrices $\{[N_k^i]_i,\ k \in \mathsf{K},\ i = 1,2\}$ have full rank.

(iii) The matrix $D_k + D_k'$ is positive definite for all $k \in \mathsf{K}$, where $D_k = \begin{bmatrix}[D_k^1]_{11} & [D_k^1]_{12}\\ [D_k^2]_{21} & [D_k^2]_{22}\end{bmatrix}$.

Here, Assumption 4.(ii) ensures that the constraint qualification conditions hold. Assumption 4.(iii) is a sufficient condition for the cost functions (39c)–(39d) to be strictly diagonally convex in the decision variables v_k at every stage $k \in \mathsf{K}$; see also Assumption 3 for CNZDG. Similar to pNZDG as defined in the previous section, using (39), we define the following parametric unconstrained linear-quadratic difference game (pLQDG) associated with CLQDG.

$$\begin{aligned}\text{pLQDG}:\ \min_{\tilde{\mathsf{u}}^i}\Big\{&\bar{J}_i(x_0,\tilde{\mathsf{u}}^1,\tilde{\mathsf{u}}^2;\{(\mathsf{w}_\tau,\theta_\tau)\}_{\tau\in\mathsf{K}})\\ &= g_K^i(x_K,\mathsf{w}_K) + \sum_{k\in\mathsf{K}_l} g_k^i(x_k,\mathsf{u}_k,\mathsf{w}_k) - \sum_{k\in\mathsf{K}} {\theta_k^i}'(M_k^i x_k + N_k^i \mathsf{w}_k + r_k^i)\Big\}, &(40a)\\ &\text{subject to } x_{k+1} = A_k x_k + B_k^1 u_k^1 + B_k^2 u_k^2,\ k \in \mathsf{K}_l. &(40b)\end{aligned}$$

For the pLQDG, due to linearity of the state dynamics and quadratic nature of the cost functions, we have the following assumption on pFS solutions and value functions:

Assumption 5. The pFS decisions of the players $\{\xi_k^{1\star}, \xi_k^{2\star}\}$ are affine functions of the state variable

$$u_k^{i\star} \equiv \xi_k^{i\star}(x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K) = E_k^i x_k + F_k^i,\ i \in \{1,2\}, \tag{41}$$

where, $E_k^i \in \mathbb{R}^{m_i \times n}$, $F_k^i \in \mathbb{R}^{m_i}$, $k \in \mathsf{K}_l$. Further, the parametric value function for player $i \in \{1,2\}$ at stage $k \in \mathsf{K}$ has the following form:

$$\begin{aligned} W_p^i\big(k, x_k; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k}^K\big) &= \tfrac{1}{2} x_k' S_k^i x_k + {s_k^i}' x_k + m_k^i \\ &+ \sum_{\tau=k}^K \Big(\tfrac{1}{2} \begin{bmatrix} \mathsf{w}_\tau \\ \theta_\tau \end{bmatrix}' \begin{bmatrix} D_\tau^i & -{N_\tau^i}' \\ N_\tau^i & 0 \end{bmatrix} \begin{bmatrix} \mathsf{w}_\tau \\ \theta_\tau \end{bmatrix} + {d_\tau^i}' \mathsf{w}_\tau - {r_\tau^i}' \theta_\tau^i \Big), \end{aligned} \tag{42}$$

where $S_k^i \in \mathbb{R}^{n\times n}$, $s_k^i \in \mathbb{R}^n$ and $m_k^i \in \mathbb{R}$.

At stage $k \in \mathsf{K}_l$, for a fixed u_k^1, the follower minimizes the objective (see (23c)).

$$g_k^2(x_k, (u_k^1, u_k^2), \mathsf{w}_k) - {\theta_k^2}'(M_k^2 x_k + N_k^2 \mathsf{w}_k + r_k^2) + W_p^2\big(k+1, A_k x_k + B_k^1 u_k^1 + B_k^2 u_k^2; \{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=k+1}^K\big).$$

If the matrix $(R_k^{22} + {B_k^2}' S_{k+1}^2 B_k^2)$ is positive definite, then for every leader's action u_k^1, the follower's objective is strictly convex in u_k^2 resulting in an unique follower optimal response at stage k as follows:

$$\bar{\mathsf{R}}_k^2(u_k^1) = -\Upsilon_k^2({B_k^2}' S_{k+1}^2 (A_k x_k + B_k^1 u_k^1) + {B_k^2}' s_{k+1}^2), \tag{43}$$

where $\Upsilon_k^2 := (R_k^{22} + {B_k^2}' S_{k+1}^2 B_k^2)^{-1}$. Now substituting the above in the leader's optimization problem (23d) and writing the first-order condition results in

$$\begin{aligned} &\big(R_k^{11} + {B_k^1}'\big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' S_{k+1}^2 + (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)\big) B_k^1\big) u_k^{1\star} \\ &\quad + {B_k^1}'\big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' S_{k+1}^2 + (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)\big) A_k x_k \\ &+ {B_k^1}'\big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' - (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 B_k^2 \Upsilon_k^2 {B_k^2}'\big) s_{k+1}^2 \\ &+ {B_k^1}'(I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' s_{k+1}^1 = 0. \end{aligned} \tag{44}$$

If the coefficient of $u_k^{1\star}$ in (44) is positive definite then the leader's objective is strictly convex in u_k^1 resulting in a unique pFS strategy for leader at stage k. Next, using (41) from Assumption 5, the follower's pFS strategy is obtained from (43) as $E_k^2 x_k + F_k^2 = -\Upsilon_k^2\big({B_k^2}' S_{k+1}^2 (A_k x_k + B_k^1 E_k^1 x_k + B_k^1 F_k^1) + {B_k^2}' s_{k+1}^2\big)$. The leader's pFS strategy is obtained by substituting $u_k^{1*} = E_k^1 x_k + F_k^1$ in (44) and equating the coefficients of x_k on both sides, as the relation has to hold true for an arbitrary x_k. The pFS strategies of the players are solved as

$$E_k^1 = -\Upsilon_k^1 {B_k^1}'\big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' S_{k+1}^2 + (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)\big) A_k, \tag{45a}$$

$$\begin{aligned} F_k^1 = &-\Upsilon_k^1 {B_k^1}'\big((I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' s_{k+1}^1 + \big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' \\ &- (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 B_k^2 \Upsilon_k^2 {B_k^2}'\big) s_{k+1}^2\big), \end{aligned} \tag{45b}$$

$$E_k^2 = -\Upsilon_k^2 {B_k^2}' S_{k+1}^2 (A_k + B_k^1 E_k^1), \tag{45c}$$

$$F_k^2 = -\Upsilon_k^2 {B_k^2}' (s_{k+1}^2 + S_{k+1}^2 B_k^1 F_k^1), \tag{45d}$$

$$\Upsilon_k^2 = (R_k^{22} + {B_k^2}' S_{k+1}^2 B_k^2)^{-1}, \tag{45e}$$

$$\Upsilon_k^1 = \big(R_k^{11} + {B_k^1}'\big({S_{k+1}^2}' B_k^2 {\Upsilon_k^2}' R_k^{12} \Upsilon_k^2 {B_k^2}' S_{k+1}^2 + (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)' S_{k+1}^1 (I - B_k^2 \Upsilon_k^2 {B_k^2}' S_{k+1}^2)\big) B_k^1\big)^{-1}. \tag{45f}$$

The above steps are summarized in the following theorem, which characterizes the pFS solution for the pLQDG.

Theorem 3. *Consider the pLQDG described by (40) with parameters $\{(\mathsf{w}_\tau, \theta_\tau)\}_{\tau=0}^K$, and let Assumption 5 hold. Define S_k^i, s_k^i and m_k^i such that the following backward recurrence equations are verified for $i \in \{1,2\}$*

$$S_k^i = Q_k^i + \sum_{j=1}^2 {E_k^j}' R_k^{ij} E_k^j + \big(A_k + \sum_{j=1}^2 B_k^j E_k^j\big)' S_{k+1}^i \big(A_k + \sum_{j=1}^2 B_k^j E_k^j\big), \tag{46a}$$

$$s_k^i = p_k^i + L_k^i \mathsf{w}_k - M_k^{i\prime}\theta_k^i + \sum_{j=1}^{2} E_k^{j\prime} R_k^{ij} F_k^j + \big(A_k + \sum_{j=1}^{2} B_k^j E_k^j\big)'\big(s_{k+1}^i + S_{k+1}^i \sum_{j=1}^{2} B_k^j F_k^j\big), \tag{46b}$$

$$m_k^i = m_{k+1}^i + \tfrac{1}{2}\sum_{j=1}^{2} F_k^{j\prime} R_k^{ij} F_k^j + \tfrac{1}{2}\big(\sum_{j=1}^{2} B_k^j F_k^j\big)' S_{k+1}^i \big(\sum_{j=1}^{2} B_k^j F_k^j\big) + \big(\sum_{j=1}^{2} B_k^j F_k^j\big)' s_{k+1}^i. \tag{46c}$$

with terminal conditions $S_K^i = Q_K^i$, $s_K^i = p_K^i + L_K^i \mathsf{w}_K - M_K^{i\,\prime}\theta_K^i$, and $m_K^i = 0$. If the matrices Υ_k^1 and Υ_k^2 defined in (45e) and (45f) are positive definite, then $\xi_k^{i\star}(x_k) = E_k^i x_k + F_k^i$ is a pFS solution for pLQDG where E_k^i and F_k^i are given by (45a)–(45d) for $i \in \{1,2\}$, $k \in \mathsf{K}_l$. The pFS state trajectory is given by

$$x_{k+1} = (A_k + \sum_{i=1}^{2} B_k^i E_k^i) x_k + \sum_{i=1}^{2} B_k^i F_k^i. \tag{47}$$

Proof. For both leader and follower, under Assumption 5, the backward recursive relations (46) follows by comparing the coefficients of state in the verification result (23b) of Lemma 2. The positive definiteness of the matrices Υ_k^2 for all $k \in \mathsf{K}_l$ ensures that the optimal response of the follower at every stage $k \in \mathsf{K}_l$ is unique. Similarly, positive definiteness of the matrices Υ_k^1 for all $k \in \mathsf{K}_l$ results in a unique parametric Stackelberg solution for leader at every stage $k \in \mathsf{K}_l$. □

5.1 FSN solution as a linear complementarity problem

In this subsection, we demonstrate that for CLQDG the elements of the set Q, as defined in (38), can be obtained as a solutions of a large-scale linear complementarity problem. To this end, we introduce some notation. Denote $\mathsf{p}_k := \mathrm{col}(p_k^1, p_k^2)$, $\mathsf{s}_k := \mathrm{col}(s_k^1, s_k^2)$, $\mathsf{B}_k := \mathrm{row}(B_k^1, B_k^2)$, $\mathsf{E}_k := \mathrm{col}(E_k^1, E_k^2)$, $\mathsf{F}_k := \mathrm{col}(F_k^1, F_k^2)$, $L_k := \mathrm{col}(L_k^1\ L_k^2)$, $\mathsf{M}_k := M_k^1 \oplus M_k^2$, and the gain matrices G_{k+1} and H_{k+1} as

$$\begin{aligned}
[G_{k+1}]_{11} :=& -\Upsilon_k^1 B_k^{1\prime}(I - B_k^2 \Upsilon_k^2 B_k^{2\prime} S_{k+1}^2)',\\
[G_{k+1}]_{21} :=& -\Upsilon_k^2 B_k^{2\prime} S_{k+1}^2 B_k^1 [G_{k+1}]_{11}\\
[G_{k+1}]_{12} :=& -\Upsilon_k^1 B_k^{1\prime}\big(S_{k+1}^{2\;\prime} B_k^2 \Upsilon_k^{2\prime} R_k^{12} \Upsilon_k^2 B_k^{2\prime} - (I - B_k^2 \Upsilon_k^2 B_k^{2\prime} S_{k+1}^2)' S_{k+1}^1 B_k^2 \Upsilon_k^2 B_k^{2\prime}\big),\\
[G_{k+1}]_{22} :=& -\Upsilon_k^2 B_k^{2\prime}(I + S_{k+1}^2 B_k^1 [G_{k+1}]_{12}),\\
[H_{k+1}]_{ii} :=& \big(E_k^{i\prime} R_k^{ii} + (A_k + \mathsf{B}_k \mathsf{E}_k)' S_{k+1}^i B_k^i\big)[G_{k+1}]_{ii}(A_k + \mathsf{B}_k \mathsf{E}_k)'\\
&+ \big(E_k^{j\prime} R_k^{ij} + (A_k + \mathsf{B}_k \mathsf{E}_k)' S_{k+1}^i B_k^j\big)[G_{k+1}]_{ji}, \quad i,j \in \{1,2\},\ i \neq j,\\
[H_{k+1}]_{ij} :=& \big(E_k^{i\prime} R_k^{ii} + (A_k + \mathsf{B}_k \mathsf{E}_k)' S_{k+1}^i B_k^i\big)[G_{k+1}]_{ij} + \big(E_k^{j\prime} R_k^{ij} + (A_k + \mathsf{B}_k \mathsf{E}_k)' S_{k+1}^i B_k^j\big)[G_{k+1}]_{jj},\\
& i,j \in \{1,2\},\ i \neq j.
\end{aligned}$$

Using the above notations, (45b)–(45d) and (46b) can be written in vector form as

$$\mathsf{F}_k = G_{k+1}\mathsf{s}_{k+1}, \tag{48}$$

$$\mathsf{s}_k = \mathsf{p}_k + \big[\mathsf{L}_k \ \ -\mathsf{M}_k'\big][\mathsf{w}_k' \ \ \theta_k']' + H_{k+1}\mathsf{s}_{k+1}, \tag{49}$$

with the terminal condition given by $\mathsf{s}_K = \mathsf{p}_K + \big[\mathsf{L}_K \ \ -\mathsf{M}_K'\big][\mathsf{w}_K' \ \ \theta_K']'$. Next, (49) can be solved as

$$\mathsf{s}_k = \sum_{\tau=k}^{K} \varphi(k,\tau)\Big(\mathsf{p}_\tau + \big[\mathsf{L}_\tau \ \ -\mathsf{M}_\tau'\big][\mathsf{w}_\tau' \ \ \theta_\tau']'\Big), \tag{50}$$

where the associated state transition matrices $\varphi(k,\tau)$ are given by $\varphi(k,\tau) = I$ for $\tau = k$, and $\varphi(k,\tau) = H_{k+1}H_{k+2}\cdots H_\tau$ for $\tau > k$. Using (48) in (47), the pFS state variable evolves according the forward linear difference equation: $x_{k+1} = \bar{A}_k x_k + \bar{B}_k \mathsf{s}_{k+1}$, with $\bar{A}_k := A_k + \mathsf{B}_k \mathsf{E}_k$ and $\bar{B}_k := \mathsf{B}_k G_{k+1}$. The solution of this linear forward difference equation for $k \in \mathsf{K}_r$ is given by

$$x_k = \phi(0,k)x_0 + \sum_{\rho=0}^{k-1} \phi(\rho+1,k)\bar{B}_\rho \mathsf{s}_{\rho+1}, \tag{51}$$

where the associated state transition matrices $\phi(\rho,k)$ are defined as $\phi(\rho,k) = I$ for $\rho = k$, and $\phi(\rho,k) = \bar{A}_{k-1}\bar{A}_{k-2}\cdots\bar{A}_\rho$ for $\rho < k$. Using (50) in (51), the pFS state trajectory for $k \in \mathsf{K}_r$ is given as follows

$$x_k = \phi(0,k)x_0 + \sum_{\tau=1}^{K}\Big(\sum_{\rho=1}^{\min\,(k,\tau)} \phi(\rho,k)\bar{B}_{\rho-1}\varphi(\rho,\tau)\Big)\big(p_\tau + [\mathsf{L}_\tau \;\; -\mathsf{M}_\tau'][\mathsf{w}_\tau' \;\; \theta_\tau']'\big). \tag{52}$$

Aggregating the variables in (52) by $\mathsf{p}_\mathsf{K} := \mathrm{col}(\mathsf{p}_k)_{k=1}^K$, $x_\mathsf{K} := \mathrm{col}(x_k)_{k=1}^K$, $\mathsf{w}_\mathsf{K} := \mathrm{col}(\mathsf{w}_k)_{k=1}^K$ and $\theta_\mathsf{K} := \mathrm{col}(\theta_k)_{k=1}^K$, the pFS state trajectory x_k, $k \in \mathsf{K}_r$ is written compactly as

$$x_\mathsf{K} = \Phi_0 x_0 + \Phi_1 \mathsf{p}_\mathsf{K} + \Phi_2 \mathsf{w}_\mathsf{K} + \Phi_3 \theta_\mathsf{K}, \tag{53}$$

where for all $k,\tau \in \mathsf{K}_r$ the matrices appearing on the right-hand side of (53) are given by $[\Phi_0]_k := \phi(0,k)$, $[\Phi_1]_{k\tau} := \sum_{\rho=1}^{\min(k,\tau)} \phi(\rho,k)\bar{B}_{\rho-1}\varphi(\rho,\tau)$, $[\Phi_2]_{k\tau} := \sum_{\rho=1}^{\min(k,\tau)} \phi(\rho,k)\bar{B}_{\rho-1}\varphi(\rho,\tau)\mathsf{L}_\tau$, $[\Phi_3]_{k\tau} := -\sum_{\rho=1}^{\min\,(k,\tau)} \phi(\rho,k)\bar{B}_{\rho-1}\varphi(\rho,\tau)\mathsf{M}_\tau'$. Next, the parametric complementarity problem (36) can be written as the following parametric linear complementarity problem (pLCP):

$$\mathrm{pLCP}(x_\mathsf{K}):\; 0 \le \begin{bmatrix}\mathsf{D}_\mathsf{K} & -\bar{\mathsf{N}}_\mathsf{K}' \\ \mathsf{N}_\mathsf{K} & 0\end{bmatrix}\begin{bmatrix}\mathsf{w}_\mathsf{K} \\ \theta_\mathsf{K}\end{bmatrix} + \begin{bmatrix}\bar{\mathsf{L}}_\mathsf{K}' \\ \bar{\mathsf{M}}_\mathsf{K}\end{bmatrix} x_\mathsf{K} + \begin{bmatrix}\mathsf{d}_\mathsf{K} \\ \mathsf{r}_\mathsf{K}\end{bmatrix} \perp \begin{bmatrix}\mathsf{w}_\mathsf{K} \\ \theta_\mathsf{K}\end{bmatrix} \ge 0, \tag{54}$$

where $\mathsf{D}_\mathsf{K} = \oplus_{k=1}^K \mathsf{D}_k$, $\mathsf{N}_\mathsf{K} = \oplus_{k=1}^K (N_k^1 \oplus N_k^2)$, $\bar{\mathsf{N}}_\mathsf{K} = \oplus_{k=1}^K([N_k^1]_1 \oplus [N_k^2]_2)$, $\bar{\mathsf{L}}_\mathsf{K} = \oplus_{k=1}^K(\mathrm{row}([L_k^1]_1,[L_k^2]_2))$, $\bar{\mathsf{M}}_\mathsf{K} = \oplus_{k=1}^K(\mathrm{col}(M_k^1,M_k^2))$, $\mathsf{d}_\mathsf{K} = \mathrm{col}(\mathrm{col}([d_k^1]_1,[d_k^2]_2))_{k=1}^K$, $\mathsf{r}_\mathsf{K} = \mathrm{col}(\mathrm{col}(r_k^1,r_k^2))_{k=1}^K$. Using (53) in (54), we obtain the following single linear complementarity problem:

$$\mathrm{LCP}: 0 \le \mathbf{M}_\mathsf{K}\begin{bmatrix}\mathsf{w}_\mathsf{K} \\ \theta_\mathsf{K}\end{bmatrix} + \mathbf{q}_\mathsf{K} \perp \begin{bmatrix}\mathsf{w}_\mathsf{K} \\ \theta_\mathsf{K}\end{bmatrix} \ge 0, \tag{55}$$

where $\mathbf{M}_\mathsf{K} := \begin{bmatrix}\mathsf{D}_\mathsf{K} + \bar{\mathsf{L}}_\mathsf{K}'\Phi_2 & -\bar{\mathsf{N}}_\mathsf{K}' + \bar{\mathsf{L}}_\mathsf{K}'\Phi_3 \\ \mathsf{N}_\mathsf{K} + \bar{\mathsf{M}}_\mathsf{K}\Phi_2 & \bar{\mathsf{M}}_\mathsf{K}\Phi_3\end{bmatrix}$ and $\mathbf{q}_\mathsf{K} := \begin{bmatrix}\mathsf{d}_\mathsf{K} + \bar{\mathsf{L}}_\mathsf{K}'\Phi_1\mathsf{p}_\mathsf{K} + \bar{\mathsf{L}}_\mathsf{K}'\Phi_0 x_0 \\ \mathsf{r}_\mathsf{K} + \bar{\mathsf{M}}_\mathsf{K}\Phi_1\mathsf{p}_\mathsf{K} + \bar{\mathsf{M}}_\mathsf{K}\Phi_0 x_0\end{bmatrix}$.

Theorem 4. *Consider the CLQDG described by (39). Let Assumptions 4 and 5 hold true and the matrices Υ_k^1 and Υ_k^2 defined in (45e) and (45f) are positive definite for all $k \in \mathsf{K}_l$. If $\mathit{SOL}(\mathrm{pLCP}_0(x_0) \neq \emptyset$ and $\mathit{SOL}(\mathrm{LCP}) \neq \emptyset$, then the FSN strategies of the players in CLQDG are given by*

$$\{\psi^{i\star} \equiv (\{(\gamma_k^{i\star}(x_k), v_k^{i\star})\}_{k\in\mathsf{K}_l}, v_K^{i\star}),\; i = 1,2\},$$

where the simultaneous decisions are $(\mathsf{v}_0^\star, \mu_0^\star) = \mathit{SOL}(\mathrm{pLCP}(x_0))$, $(\mathsf{v}_\mathsf{K}^\star, \mu_\mathsf{K}^\star) = \mathit{SOL}(\mathrm{LCP})$ and the sequential decisions are $\gamma_k^{i\star}(x_k) = E_k^i x_k + F_k^i$, $i \in \{1,2\}$, where E_k^i and F_k^i, $i \in \{1,2\}$ are given by (45) with parameters as $(\mathsf{v}_\mathsf{K}^\star, \mu_\mathsf{K}^\star)$.

Proof. The proof follows from Proposition 1 and the steps before the theorem. □

Remark 15. The LCP (55) may not have a solution, and if it does, there could be one, more than one, or a continuum of equilibrium solutions. In cases where the LCP has multiple solutions, each of these solutions will result in FSN strategies where the leader can enforce her sequential decision locally at each stage. Furthermore, following Remark 13, the leader can order the FSN solutions in terms of her global cost; see Remark 9. The conditions for the existence and uniqueness of LCP solutions, along with the numerical methods to obtain them, have been extensively studied in the optimization literature. For more details on this topic, see [14].

6 Numerical illustration

In this section, we illustrate our results with a numerical example. The setting is the same as in [32], i.e., two firms compete in the same market in quantities (i.e., à la Cournot) and invest in process R&D to reduce their unit production costs, and in their production capacity. The difference with [32] is that here the firms announce sequentially, and not simultaneously, their investments. As in the motivating example in the introduction, we suppose that player 1 is an international company that acts as leader in the investment variables and player 2 is a local firm that acts as follower. Denote by v_k^i the quantity produced (output) by firm $i \in \{1,2\}$ at time k. The price $P(v_k^1, v_k^2)$ is given by the following affine inverse demand:

$$P(v_k^1, v_k^2) = \bar{A}_k - \bar{B}_k(v_k^1 + v_k^2), \quad k \in \mathsf{K}.$$

We assume that $\bar{A}_k = \bar{A}_{k-1}(1+\epsilon)$ and $\bar{B}_k = \bar{B}_{k-1}/(1+\epsilon)$, with $\epsilon > 0$ for all $k \in \mathsf{K}_r$, that is, the demand increases over time. The unit-production cost of each firm is decreasing in its stock of R&D, X_k^i, whose evolution is described by the following difference equation:

$$X_{k+1}^i = \mu^i X_k^i + R_k^i + \lambda^i R_k^j, \quad i \neq j, \quad i,j \in \{1,2\}, \tag{56a}$$

where R_k^i, denotes the investment in R&D by firm i at time $k \in \mathsf{K}_l$. The spillover parameter $\lambda^i \in (0,1)$ represents the part of firm i's investment in R&D that spills over to its rival. This means that knowledge by a firm is not fully appropriable by the investor. Let I_k^i be the investment made by firm $i \in \{1,2\}$ to increase its production capacity, Y_k^i, whose evolution is given by

$$Y_{k+1}^i = \delta^i Y_k^i + I_k^i, \quad i \in \{1,2\}, \tag{56b}$$

where $(1-\delta^i)$ represents the depreciation rate. Each firm's production must be non-negative and is upper bounded by its production capacity, i.e.,

$$Y_k^i \geq v_k^i \geq 0, \quad i \in \{1,2\}. \tag{57}$$

Production decisions are taken simultaneously. The production cost of firm i is given by $h_i(X_k^i, v_k^i) = (c^i - \gamma^i X_k^i)v_k^i$. Here, c_i is the initial fixed cost, γ^i is the positive cost learning parameter that represents the speed of reduction in the unit cost. The investment costs in R&D and production capacity are given by $g_i(R_k^i) = (a^i/2)(R_k^i)^2$ and $f_i(I_k^i) = (b^i/2)(I_k^i)^2$, respectively, where a^i and b^i are positive parameters. At terminal time K, the salvage value is given by $S_i(X_K^i, Y_K^i) = (\alpha_X^i/2)(X_K^i)^2 + (\alpha_Y^i/2)(Y_K^i)^2$. Firm i's objective is given by

$$J^i = \sum_{k=0}^{K-1} \beta^k (f_i(R_k^i) + g_i(I_k^i)) + \sum_{k=0}^{K} \beta^k (h_i(X_k^i, v_k^i) - P(v_k^1, v_k^2)v_k^i) + \beta^K S_i(X_K^i, Y_K^i), \tag{58}$$

where β is the common discount factor. Each firm $i \in \{1,2\}$ minimizes its objective (58) subject to the state dynamics (56) and the production capacity constraint (57). Note that the dynamic duopoly game (56)–(58) fits into the CLQDG (39). For numerical illustration, we assume the following parameters: $K = 14$, $\bar{A}_0 = 3.5$, $\bar{B}_0 = 0.5$, $\epsilon = 0.015$, $\beta = 0.9$, $\lambda^i = 0.1$, $\mu^i = 0.8$, $\delta^i = 0.85$, $\gamma^i = 0.2$, $a^i = 1$, $b^i = 1$, $c^i = 0.5$, $\alpha_X^i = -0.2$, $\alpha_Y^i = -0.25$, $X_0^i = 5$, $Y_0^i = 4$, $i \in \{1,2\}$. We used the freely available PATH solver (available at `https://pages.cs.wisc.edu/ferris/path.html`) for solving the LCP (55).

Figure 3 illustrates the evolution of the stock of knowledge, production capacity and the quantities produced by both firms under the FSN solution. Figure 3a and 3b in top panel, show the evolution of production capacity along with the outputs of the foreign firm (leader) and the local firm (follower), respectively. For the case when the investment made by both firms are also simultaneous (for example when both are local firms), the corresponding plots for the evolution of production capacity and the outputs are obtained from feedback-Nash equilibria using the approach in [32] and are shown in Figure 3c. In all cases, we note that, the outputs are always upper bounded by the production capacities of the firms. Also, in each case, the constraints are active from time period 4 to 13. Bottom panel in Figure 3 shows the comparison between the leader, follower and feedback Nash (when the investment made by both firms are also simultaneous). Even though the initial values of the stock of knowledge and production capacity of both firm are the same, as time progress, from Figure 3d and Figure 3e, we observe that, both stocks of knowledge and production capacity, are higher for the foreign firm (leader) as compared to the local firm (follower). Also, the quantity of goods produced by the foreign firm are consistently higher as compared to the local firm; see Figure 3f. In all the bottom plots in Figure 3, we also notice that the feedback-Nash equilibria plots lie in between the leader and follower plots.

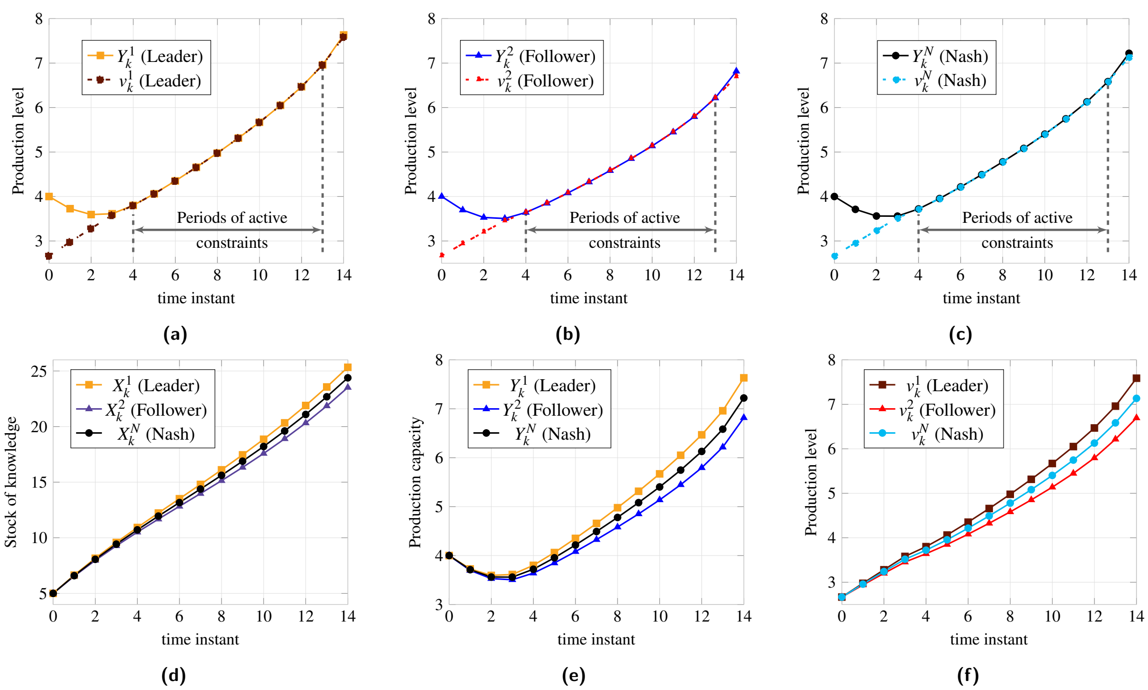

Figure 3: Evolution of production capacities and production quantities of foreign firm/leader (panel (a)), local firm/follower (panel (b)) and each firm when all decisions are simultaneous/Nash (panel (c)). Comparison of evolution of stock of knowledge (panel (d)), production capacities (panel (e)) and production quantity (panel (f)) of foreign firm (leader), local firm (follower) and each firm when all decisions are simultaneous (Nash).

7 Conclusions

We studied a class of two-player nonzero-sum difference games with coupled inequality constraints, where players interact sequentially in one type of decision variables and simultaneously in the other type of decision variables. For this quasi-hierarchical interaction, we defined the feedback Stackelberg-Nash (FSN) solution and provided a recursive formulation of this solution under separability assumption of cost function. We showed that the FSN solution of these constrained games can be derived using the parametric feedback Stackelberg solution of an associated unconstrained parametric game involving only sequential decision variables, with a specific choice of the parameters, that satisfy some implicit complementarity conditions. We specialized these results to a linear-quadratic setting involving affine inequality constraints, and for this special case, we showed that the FSN solution can be obtained from the solution of a large-scale linear-complementarity problem.

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