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Classical solutions to graphon MFG equations with affine control: Lipschitz mappings on Hölder spaces

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Abstract : The solution of a graphon mean field game (GMFG) is characterized by a Hamilton-Jacobi-Bellman (HJB) equation and a Fokker-Planck-Kolmogorov (FPK) equation linked together via a graphon coupling function. We analyze the classical solution of the GMFG equation system on Hölder spaces. We study the best response control problem and specify the operator that regenerates the graphon coupling function. This operator is shown to be a Lipschitz mapping and is contractive under some conditions, which leads to the existence and uniqueness of the solution of the GMFG equation system.

Résumé : La solution de l'équation du jeu de champ moyen du graphon (GMFG) est caractérisée par une équation HJB et une équation FPK couplées via un champ moyen de graphon. Une analyse de la solution classique du système d'équations GMFG sur des espaces Holder sont prévus. Dans ce cadre, une solution au problème de contrôle de la meilleure réponse est dérivée, donnant un opérateur qui régénère le terme de champ moyen du graphon. Cet opérateur se révèle être une application Lipschitzienne contractive dans des conditions spécifiées qui donnent par conséquent l'existence et l'unicité de la solution du système d'équations GMFG.

Mots clés : Grands réseaux, jeux de champ moyen

1 Introduction and the infinite population model

Graphon mean field games provide a significant generalization of the standard mean field game framework [15, 19] by incorporating heterogeneity of spatially distributed agents. General nonlinear GMFG models have been introduced in the previous work [5]. For further references, see [3, 13].

Consider the state equation of a representative agent at node α (to be called the α-agent):

$$dX_t^\alpha = [a(X_t^\alpha) + bu_t + c(X_t^\alpha)z^\alpha(t)]dt + \sqrt{2}dW_t^\alpha, \tag{1}$$

where $X_t^\alpha \in \mathbb{R}$ is the state, $u_t \in \mathbb{R}$ the control, and $W_t^\alpha \in \mathbb{R}$ a standard Brownian motion. The initial state X_0^α has probability density function $p^\alpha(x)$. The control gain b is nonzero. For simplicity, we consider a scalar state X_t^α, and will discuss later the extension to the vector state case. The graphon network coupling term is given by

$$z^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\mu^\beta(t,dx)d\beta, \tag{2}$$

where $g : [0,1]^2 \to [0,1]$ is the graphon function, and $\mu^\beta(t,dx)$ the distribution of X_t^β. The averaging in the right hand side of (2) is based on a function χ of the state. In the subsequent analysis, z as a function of (t,α) will be called the graphon coupling function. The cost of the α-agent is

$$J^\alpha = E\int_0^T \Big[L(t, X_t^\alpha, z^\alpha(t)) + ru_t^2\Big]dt, \tag{3}$$

where the control penalty parameter $r > 0$ is a constant.

The above graphon interaction model may be interpreted according to the limit of a finite population of agents distributed over dense networks. Consider a network of vertices $\{1/N, \cdots, (N-1)/N, 1\}$. The agent at node i/N has the state process

$$dX_t^i = [a(X_t^i) + bu_t^i + c(X_t^i)z^{N,i}(t)]dt + \sqrt{2}dW_t^i, \quad 1 \le i \le N,$$

where the coupling term $z^{N,i}$ is given by

$$z^{N,i}(t) = \frac{1}{N}\sum_{j=1}^N g_{ij}^N \chi(X^j(t)).$$

Similarly, an individual cost can be specified using $L(t, X_t^i, z^{N,i}(t))$. The $N \times N$ dimensional matrix $g^N := (g_{ij}^N)_{1\le i,j\le N}$ is symmetric with $g_{ij}^N \in [0,1]$, and is interpreted as the adjacency matrix of an undirected graph. The matrix g^N may be represented as a step function defined on the unit square $[0,1]^2$. Suppose when $N \to \infty$, the sequence of step functions converges in a suitable sense to the graphon g while i/N approaches $\alpha \in [0,1]$. Subsequently, $z^{N,i}(t)$ is further approximated by $z^\alpha(t)$ in (2). Accordingly, X_t^i is approximated by X_t^α which is labelled by α taken from the continuum $[0,1]$ as the vertex set.

Our previous work [5] introduces a general nonlinear GMFG model with control taking its value from a compact set, and the existence and uniqueness of a solution is established if certain parameters fulfill a contraction condition when the graphon weighted mean field term is iterated. It further establishes an ϵ-Nash equilibrium property for the resulting decentralized strategies applied by a large but finite population. For a GMFG model with affine dynamics, the existence and uniqueness of a solution is established in [3] for the graphon coupled HJB-FPK equation system by extending the Schauder fixed point method with symmetric players in [8]. More recently, ref. [13] analyzes a linear-quadratic GMFG and develops subspace-based numerical computation techniques. The reader may refer to [2] for the analysis of stochastic mean field dynamics with graphon coupling, [9, 22] for static graphon games, [17] for general dynamic games with agents distributed over sparse networks, [6, 7, 11] for extension of graphon mean field game modeling to sparse networks, and [23] for learning algorithms in graphon mean field games.

Note that the contraction condition in [5] is generally difficult to verify while [3] requires the state to be from a torus. In this paper we allow the state lying in a Euclidean space. By considering nonlinear state dynamics and linear control with quadratic penalty, we will be able to exploit the analytical property of the operator governing the mean field iterations. More specifically, we will prove a Lipschitz property of such a mapping, which so far has not been well explored in the literature. In our current setup, we may identify nontrivial models to verify the contraction condition and get existence and uniqueness without the monotonicity condition for MFGs [8] and GMFGs [3].

1.1 The best response control problem

Let $z^\alpha(\cdot)$ be fixed. The α-agent solves its best response control problem with dynamics and cost given by (1) and (3). Let $V^\alpha(t,x)$ denote the value function of the α-agent. The HJB equation takes the form:

$$\begin{aligned} 0 =& V_t^\alpha(t,x) + V_{xx}^\alpha + \min_{u\in\mathbb{R}} \Big\{ V_x^\alpha[a(x) + bu + c(x)z^\alpha(t)] + ru^2 \Big\} \\ &+ L(t,x,z^\alpha(t)), \end{aligned} \tag{4}$$

where $V^\alpha(T,x) = 0$. The minimizer is $\hat{u} = -bV_x^\alpha/(2r)$. Denote

$$b_0 = \frac{b^2}{4r}.$$

Equation (4) is written as

$$\begin{cases} 0 = V_t^\alpha + V_{xx}^\alpha + V_x^\alpha[a(x) + c(x)z^\alpha(t)] - b_0(V_x^\alpha)^2 + L(t,x,z^\alpha(t)), \\ V^\alpha(T,x) = 0. \end{cases} \tag{5}$$

Given the control law $u = -bV_x^\alpha/(2r)$, we have the closed-loop state process

$$dX_t^\alpha = [a(X_t^\alpha) - 2b_0V_x^\alpha(t,X_t^\alpha) + c(X_t^\alpha)z^\alpha(t)]dt + \sqrt{2}dW_t^\alpha. \tag{6}$$

Let $m^\alpha(t,x)$ denote the probability density of X_t^α. The FPK equation of m^α is given by

$$\begin{cases} m_t^\alpha = m_{xx}^\alpha - \partial_x\big\{ m^\alpha(t,x)[a(x) - 2b_0V_x^\alpha(t,x) + c(x)z^\alpha(t)]\big\}, \\ m^\alpha(0,x) = p^\alpha(x). \end{cases} \tag{7}$$

In the derivation of the above HJB equation and FPK equation, we have assumed that z^α is given. To find a solution to the GMFG, we need to determine z^α by imposing condition (2).

1.2 The GMFG equation system

The solution of the GMFG is described by the equation system:

$$0 = V_t^\alpha + V_{xx}^\alpha + V_x^\alpha[a(x) + c(x)z^\alpha(t)] - b_0(V_x^\alpha)^2 + L(t,x,z^\alpha(t)), \tag{8}$$

$$m_t^\alpha = m_{xx}^\alpha - \partial_x\big\{ m^\alpha(t,x)[a(x) - 2b_0V_x^\alpha(t,x) + c(x)z^\alpha(t)]\big\}, \tag{9}$$

where $V^\alpha(T,x) = 0$ and $m^\alpha(0,x) = p^\alpha(x)$, $\alpha \in [0,1]$ and

$$z^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)m^\beta(t,x)dxd\beta. \tag{10}$$

We call (10) the consistency condition, where the right hand side is interpreted as the graphon weighted nonlinear average of the states of all agents distributed over the network.

Our existence analysis will employ a fixed point argument. We regard $z^\alpha(t)$ as a continuous function of two variables $(t,\alpha) \in [0,T]\times[0,1]$, and call it the graphon coupling function. Given z from a suitably selected set $\mathcal{Z}$ (to be specified subsequently), for each α, we solve Equation (5) and determine the feedback control law $\hat{u}$ for the α-agent. Next, we obtain m^α from the FPK Equation (7). Finally, we determine $z_1 \in \mathcal{Z}$ by the rule:

$$z_1^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)m^\beta(t,x)dxd\beta, \quad \forall \alpha \in [0,1], \tag{11}$$

which is equivalently written using an operator Φ:

$$z_1 = \Phi z.$$

Note that the right hand side of (11) depends on z, which has been used to determine $(V^\alpha)_{0\le\alpha\le 1}$ and subsequently $(m^\beta(t,x))_{0\le\beta\le 1}$. So the GMFG solution is formulated as solving the fixed point problem

$$z = \Phi z, \quad z \in \mathcal{Z}.$$

We make the following assumptions:

(A1) The functions $a(x)$, $a_x(x)$, $c(x)$, and $c_x(x)$ are bounded continuous functions, and a_x, c_x are both in the Hölder space $C^\gamma(\mathbb{R})$ with Hölder exponent $\gamma \in (0,1)$.

(A2) L is nonnegative, bounded and continuous in $(t,x,z) \in [0,T]\times\mathbb{R}^2$, and

$$\sup_{t,x,z} L(t,x,z) \le L_0.$$

The partial derivatives $L_t, L_x, L_z, L_{zt}, L_{zx}, L_{zz}$ exist and are bounded and continuous on $[0,T]\times\mathbb{R}^2$.

(A3) The initial probability density function $p^\alpha(x)$ is continuous in $(\alpha,x) \in [0,1]\times\mathbb{R}$ and $p^\alpha(\cdot) \in C^{2+\gamma}(\mathbb{R})$.

(A4) χ is bounded, Lipschitz continuous (with Lipschitz constant $\mathrm{Lip}(\chi)$), and

$$\int_{\mathbb{R}} |\chi(x)|dx =: C_\chi < \infty.$$

(A5) $g : [0,1]^2 \to [0,1]$ is a measurable function, and g maps $C([0,1])$ to $C([0,1])$, i.e., given $h \in C([0,T])$, the mapping

$$\alpha \mapsto \int_0^1 g(\alpha,\beta)h(\beta)d\beta, \quad \alpha \in [0,1],$$

is a continuous function defined on $[0,1]$.

In the above, we use $C([0,T])$ to denote the set of $\mathbb{R}$-valued continuous functions defined on $[0,T]$. The Hölder spaces $C^\gamma(\mathbb{R})$ and $C^{2+\gamma}(\mathbb{R})$ are defined below.

1.3 Notation

If the function $h(x)$ is defined on a set $Q \subset \mathbb{R}^n$, we denote the norm $|h|_{0;Q} = \sup_{x\in Q}|g(x)|$ and the Hölder semi-norm $[h]_{\gamma;Q} = \sup_{x,x'} |h(x)-h(x')|/|x-x'|^\gamma$ for $\gamma \in (0,1)$. If $f(t,x)$ is defined on the set $Q_T = [0,T]\times Q$, define the Hölder semi-norms (see [16])

$$[f]_{\gamma/2,\gamma;Q_T} = \sup_{(t,x),(s,y)\in Q_T} \frac{|f(t,x)-f(s,y)|}{(|t-s|^{1/2}+|x-y|)^\gamma},$$

and

$$[f]_{1+\gamma/2,2+\gamma,Q_T} = [f_t]_{\gamma/2,\gamma;Q_T} + \sum_{i,j}[f_{x_ix_j}]_{\gamma/2,\gamma;Q_T}.$$

Denote the Hölder norms

$$\begin{aligned}|h|_{\gamma;Q} =& |h|_{0;Q} + [h]_{\gamma;Q},\\ |f|_{\gamma/2,\gamma;Q_T} =& |f|_{0;Q_T} + [f]_{\gamma/2,\gamma;Q_T},\\ |f|_{1+\gamma/2,2+\gamma;Q_T} =& |f|_{0;Q_T} + |f_t|_{0;Q_T} + \sum_i |f_{x_i}|_{0;Q_T}\\ &+ \sum_{i,j} |f_{x_ix_j}|_{0;Q_T} + [f]_{1+\gamma/2,2+\gamma;Q_T}.\end{aligned}$$

The subscript Q or Q_T in the norm/semi-norm may be omitted if it is clear from the context. The Hölder space $C^{\gamma/2,\gamma}(Q_T)$ (resp., $C^{1+\gamma/2,2+\gamma}(Q_T)$) consists of all functions with $|f|_{\gamma/2,\gamma;Q_T} < \infty$ (resp., $|f|_{1+\gamma/2,2+\gamma;Q_T} < \infty$). The Hölder space $C^{2+\gamma}(Q)$ is similarly defined with the norm $|h|_{2+\gamma;Q} = |f|_{0;Q} + \sum_i |f_{x_i}|_{0;Q} + \sum_{i,j} |f_{x_ix_j}|_{0;Q} + \sum_{i,j}[f_{x_ix_j}]_{\gamma;Q}$. We will solve the HJB Equation (5) and the FPK Equation (7) in the Hölder space $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$. We consider the case with x being a scalar. The general case may be treated similarly.

2 The HJB equation and FPK equation with a given z

Consider the following parabolic equation

$$\partial_t u(t,x) - \varDelta u(t,x) + \langle a_1(t,x), \partial_x u(t,x)\rangle + a_0(t,x)u(t,x) = f(t,x), \tag{12}$$

where $\varDelta$ is the Laplacian operator, $u(0,x) = \psi(x)$, and $t \in [0,T]$, $x \in \mathbb{R}^n$. The function a_1 is $\mathbb{R}^n$-valued with its k-th component denoted by $a_{1,k}$.

Theorem 1. ([18, 21]) *Suppose $a_{1,k}, a_0, f \in C^{\gamma/2,\gamma}([0,T]\times\mathbb{R}^n)$, and $\psi \in C^{2+\gamma}(\mathbb{R}^n)$. Then Equation (12) has a unique solution u from the class $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R}^n)$ and for some constant K_0,*

$$|u|_{1+\gamma/2,2+\gamma} \le K_0(|f|_{\gamma/2,\gamma} + |\psi|_{2+\gamma}). \tag{13}$$

Remark 1. When the coefficients a_0 and a_1 are allowed to change within two given sets, the constant K_0 can be selected depending only on the upper bound of the Hölder norms of a_0 and a_1.

2.1 The Hopf-Cole transformation

Fix α and consider $z^\alpha(\cdot)$ as a Hölder continuous function of $t \in [0,T]$. We apply the Hopf-Cole transformation $w = e^{-b_0V^\alpha}$ with $b_0 = b^2/(4r)$ and rewrite Equation (5) in the following form:

$$0 = w_t + w_{xx} + w_x[a(x) + c(x)z^\alpha(t)] - b_0 w L(t,x,z^\alpha(t)), \qquad (t,x) \in (0,T)\times\mathbb{R}, \tag{14}$$

where $w(T,x) = 1$.

Theorem 2. *Suppose that Assumptions (A1)–(A2) hold with $a_x, c_x \in C^\gamma(\mathbb{R})$, and that $z^\alpha \in C^{\gamma/2}([0,T])$ is given. Then the following holds: (i) Equation (14) has a unique solution w in the class $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$ and moreover $e^{-b_0L_0T} \le w \le 1$; (ii) Equation (5) has a unique solution V^α in the class $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$.*

Proof. Under Assumption (A1), $a(x) + c(x)z^\alpha(t)$ and $L(t,x,z^\alpha(t))$ are both in $C^{\gamma/2,\gamma}([0,T]\times\mathbb{R})$. By Theorem 1 we obtain a unique solution w for (14) from the class $C^{1+\gamma/2,2+\gamma}([0,T]\times\mathbb{R})$. Moreover, by the maximum principle of the Cauchy problem [18, Chapter II, Theorem 2.5], we can show

$$|w(t,x)| \le \sup_x |w(T,x)| = 1, \qquad \forall t \in [0,T], x \in \mathbb{R}. \tag{15}$$

Note that Equation (14) alone does not immediately yield the property $w > 0$, which is needed for using the transformation $V^\alpha = -b_0^{-1} \ln w$ to determine a solution of (5). Below we will develop an iterative procedure to construct a solution for Equation (5). The idea is to repeatedly raise a conservative lower bound of w so that the magnitude of the lower bound is maintained.

Step 1. By Theorem 1, for some fixed constant C_0, we have

$$|w|_{1+\gamma/2,2+\gamma} \le C_0 |w(T,\cdot)|_{2+\gamma} = C_0. \tag{16}$$

Take some fixed $\eta_0 < T$ such that

$$\frac{e^{-b_0 L_0 T}}{2|\eta_0|} \ge C_0 + 1. \tag{17}$$

We claim that $w(t,x) \ge \frac{1}{2} e^{-b_0 L_0 T}$ holds for all $t \in [T-\eta_0, T]$ and $x \in \mathbb{R}$. Otherwise, by using the terminal condition at T and by the mean value theorem, there would exist at least one point (t_0, x_0) with $t_0 \in [T-\eta_0, T]$ such that $|w_t(t_0, x_0)| \ge C_0 + 1$, which contradicts (16).

Now on $[T-\eta_0, T] \times \mathbb{R}$, we take $V^\alpha(t,x) = -b_0^{-1} \ln w(t,x)$. Accordingly, with $t \ge T - \eta_0$, we get boundedness of V^α, V^α_t, V^α_x and V^α_{xx} on $[T-\eta_0, T] \times \mathbb{R}$.

Step 2. Given the boundedness of V^α and of its derivatives on $[T-\eta_0, T] \times \mathbb{R}$, we may interpret V^α as the value function of an optimal control problem (see e.g. [12, Chapter VI]) that has dynamics (1) and cost (3) redefined on time horizon $[T-\eta_0, T]$. Hence $0 \le V^\alpha(t,x) \le L_0 T$ for $t \in [T-\eta_0, T]$. (We do not attempt to make the upper bound tight.) Subsequently, we have the updated estimate $e^{-b_0 L_0 T} \le w(t,x) \le 1$ for $t \in [T-\eta_0, T]$, $x \in \mathbb{R}$.

Step 3. Consider $[T-2\eta_0, T]$. We similarly have $\frac{1}{2} e^{-b_0 L_0 T} \le w(T-t, x) \le 1$ for all $t \in [T-2\eta_0, T-\eta_0]$. Then by relating to an optimal control problem on $[T-2\eta_0, T]$ as in step 2, we show $w(t,x) \ge e^{-b_0 T L_0}$ for $t \in [T-2\eta_0, T-\eta_0]$. After a finite number of iterations, we can cover the whole interval $[0,T]$, where the last step treats an interval of the form $[0, T-k\eta_0]$ with $0 < T - k\eta_0 \le \eta_0$. Finally, we conclude that $e^{-b_0 L_0 T} \le w(t,x) \le 1$ for all $t \in [0,T]$ and $x \in \mathbb{R}$. This accordingly determines a solution $V^\alpha = -b_0^{-1} \ln w$ for (5) on $[0,T] \times \mathbb{R}$. The solution V^α from the class $C^{1+\gamma/2,2+\gamma}([0,T] \times \mathbb{R})$ is unique by the uniqueness result of w. □

Remark 2. If the model has a vector state $X_t^\alpha \in \mathbb{R}^n$, and u_t is $\mathbb{R}^{n_1}$-valued, the Hopf-Cole transformation still works as long as we take the control as bu_t with $b \in \mathbb{R}$, but will not work if b is replaced by a general matrix B.

2.2 Solution of the FPK equation

Let V^α be given by Theorem 2. We rewrite (7) in the form

$$\begin{aligned} m_t^\alpha =& m_{xx}^\alpha - m_x^\alpha [a(x) - 2b_0 V_x^\alpha(t,x) + c(x) z^\alpha(t)] \\ &- m^\alpha \partial_x [a(x) - 2b_0 V_x^\alpha(t,x) + c(x) z^\alpha(t)], \end{aligned} \tag{18}$$

which is a linear equation with coefficients in the Hölder space $C^{\gamma/2,\gamma}([0,T] \times \mathbb{R})$. The following proposition results from Theorem 1.

Proposition 1. *Under Assumptions (A1), (A2) and (A3), for Equation (7) there exists a unique solution m^α from the class $C^{1+\gamma/2,2+\gamma}([0,T] \times \mathbb{R})$.* □

2.3 A priori gradient estimate in (5)

Although Theorem 2 shows that $|V_x^\alpha|$ is bounded, it does not give an explicit upper bound in terms of parameters and bounds of known functions in (5). We will estimate the x-gradient of the value function using a comparison argument ([12, Appendix E], [8]). Let $J^\alpha(t,x,u(\cdot))$ be the cost with initial condition (t,x) on $[t,T]$ in place of (3). Denote

$$V^\alpha(t,x) = J^\alpha(t,x,u^x), \qquad V^\alpha(t,y) = J^\alpha(t,y,u^y).$$

Here $u^x(s,\omega) := -V_x^\alpha(s,X_s^\alpha)/(2r)$, $s\in[t,T]$, is the progressively measurable control process generated by the closed-loop dynamics

$$dX_s^\alpha = [a(X_s^\alpha) - 2b_0 V_x^\alpha(s,X_s^\alpha) + c(X_s^\alpha)z^\alpha(s)]ds + \sqrt{2}dW_s^\alpha, \qquad s\ge t, \tag{19}$$

where $X_t^\alpha = x$. Without loss of generality, suppose $V^\alpha(t,x)\le V^\alpha(t,y)$. Note that $u^x(s,\omega)$ is suboptimal for the control problem with initial condition (t,y). Then

$$|V^\alpha(t,x) - V^\alpha(t,y)| \le |J^\alpha(t,x,u^x) - J^\alpha(t,y,u^x)|. \tag{20}$$

Now we consider the two state processes

$$dX_s^x = [a(X_s^x) + bu^x(s,\omega) + c(X_s^x)z^\alpha(s)]ds + \sqrt{2}dW_s,$$
$$dX_s^y = [a(X_s^y) + bu^x(s,\omega) + c(X_s^y)z^\alpha(s)]ds + \sqrt{2}dW_s,$$

where $X_t^x = x$ and $X_t^y = y$ and the same Brownian motion W_s is used. Note that the GMFG equation system will eventually be solved subject to condition (2). Here we consider a general function $z^\alpha(\cdot)\in C^{\gamma/2}([0,T])$ by merely requiring $\sup_{0\le t\le T}|z^\alpha(t)|\le|\chi|_0$, which is relaxed from (2). We use Grönwall's inequality to show

$$|X_s^x - X_s^y| \le C_T^*|x-y|,$$

with $C_T^* := \exp([|a_x|_0 + |c_x|_0\cdot|\chi|_0]T)$. We further use (3) with initial time t and the Lipschitz continuity of L to get the bound

$$|J^\alpha(t,x,u^x) - J^\alpha(t,y,u^x)| \le \mathrm{Lip}_x(L)C_T^*T|x-y|, \tag{21}$$

where $\mathrm{Lip}_x(L) := \sup_{t,x,z}|L_x(t,x,z)|$. Therefore, it follows from (20) and (21) that

$$|V_x^\alpha| \le \mathrm{Lip}_x(L)C_T^*T =: C_1^*.$$

Remark 3. If b depends on x, the above method of gradient estimates does not work.

Remark 4. The bound of the gradient does not depend on r.

2.4 Selection of the set $\mathcal{Z}$

We need to specify a set $\mathcal{Z}$ for z. Recall that we have

$$dX_t^\alpha = [a(X_t^\alpha) - 2b_0V_x^\alpha(t,X_t^\alpha) + c(X_t^\alpha)z^\alpha(t)]dt + \sqrt{2}dW_t^\alpha, \qquad 0\le t\le T.$$

By Assumption (A4), we have

$$E|\chi(X_t^\alpha)| \le |\chi|_0.$$

Now we consider z satisfying $\sup_{t,\alpha}|z^\alpha(t)|\le|\chi|_0$. Since

$$E|X_t^\alpha - X_s^\alpha| = E\int_s^t |a(X_\tau^\alpha) - 2b_0V_x^\alpha(\tau,X_\tau^\alpha) + c(X_\tau^\alpha)z^\alpha(\tau)|d\tau + \sqrt{2}E|W_t^\alpha - W_s^\alpha|,$$

it follows that

$$\begin{aligned} E|\chi(X_t^\alpha) - \chi(X_s^\alpha)| &\le \mathrm{Lip}(\chi) E|X_t^\alpha - X_s^\alpha| \\ &\le \mathrm{Lip}(\chi)[C_2^*|t-s| + \sqrt{2}|t-s|^{1/2}], \end{aligned}$$

where $C_2^* := |a|_0 + 2b_0 C_1^* + |c|_0 \cdot |\chi|_0$, for all $\alpha \in [0,1]$.

We take $\gamma \in (0,1)$ as in Assumption (A1). Then

$$\frac{E|\chi(X_t^\alpha) - \chi(X_s^\alpha)|}{|t-s|^{\gamma/2}} \le \mathrm{Lip}(\chi)(C_2^* T^{1-\gamma/2} + \sqrt{2} T^{(1-\gamma)/2}) := C_3^*. \tag{22}$$

Define

$$z_1^\alpha(t) = \int_0^1 g(\alpha,\beta) E\chi(X_t^\beta) d\beta.$$

By (22), we have

$$\begin{aligned} |z_1^\alpha(t) - z_1^\alpha(s)| &= |\int_0^1 g(\alpha,\beta) |E\chi(X_t^\beta) - E\chi(X_s^\beta)| d\beta \\ &\le C_3^* |t-s|^{\gamma/2}. \end{aligned}$$

We need to choose $\mathcal{Z}$ to ensure that z_1 remains in $\mathcal{Z}$.

Now we are ready to specify the following set $\mathcal{Z}$ consisting of all z satisfying the two conditions: (i) z a continuous function of (t,α) defined on $[0,T] \times [0,1]$; (ii)

$$|z^\alpha(t)| \le |\chi|_0, \quad |z^\alpha(t) - z^\alpha(s)| \le C_3^* |t-s|^{\gamma/2}, \quad \forall t,s \in [0,T],\ \alpha \in [0,1]. \tag{23}$$

In all subsequent analysis, we always consider $\mathcal{Z}$ satisfying the above conditions (i) and (ii).

3 The sensitivity analysis

Throughout this section we suppose that Assumptions (A1), (A2), (A3) and (A4) hold.

3.1 The HJB equation

For $z, \hat{z} \in \mathcal{Z}$, let V^α and $\hat{V}^\alpha$ be solved from (5) using z^α and $\hat{z}^\alpha$, respectively. Applying the Hopf-Cole transformation

$$w = e^{-b_0 V^\alpha}, \quad \hat{w} = e^{-b_0 \hat{V}^\alpha},$$

we derive two equations

$$\begin{aligned} 0 &= w_t + w_{xx} + w_x[a(x) + c(x) z^\alpha(t)] - b_0 w L(t,x,z^\alpha(t)), \\ 0 &= \hat{w}_t + \hat{w}_{xx} + \hat{w}_x[a(x) + c(x)\hat{z}^\alpha(t)] - b_0 \hat{w} L(t,x,\hat{z}^\alpha(t)), \end{aligned}$$

where $w(T,x) = \hat{w}(T,0) = 1$.

For fixed α, we view z^α and $\hat{z}^\alpha$ as two functions in $C^{\gamma/2}([0,T])$.

Lemma 1. *For some constant C_4^*, we have*

$$|w - \hat{w}|_{1+\gamma/2, 2+\gamma; Q_T} \le C_4^* |z^\alpha - \hat{z}^\alpha|_{\gamma/2;[0,T]}$$

for all $\hat{z}, z \in \mathcal{Z}$, where $Q_T = [0,T] \times \mathbb{R}$.

Proof. We write

$$\begin{aligned}0 =&\hat{w}_t + \hat{w}_{xx} + \hat{w}_x[a(x) + c(x)z^\alpha(t)] - b_0\hat{w}L(t, x, z^\alpha(t))\\&+ \hat{w}_x c(x)(\hat{z}^\alpha(t) - z^\alpha(t)) - b_0\hat{w}[L(t, x, \hat{z}^\alpha(t)) - L(t, x, z^\alpha(t))].\end{aligned}$$

Define $\phi = w - \hat{w}$. Then we have

$$\begin{aligned}0 =&\phi_t + \phi_{xx} + \phi_x[a(x) + c(x)z^\alpha(t)] - b_0\phi L(t, x, z^\alpha(t))\\&- \hat{w}_x c(x)(\hat{z}^\alpha(t) - z^\alpha(t)) + b_0\hat{w}[L(t, x, \hat{z}^\alpha(t)) - L(t, x, z^\alpha(t))].\end{aligned} \tag{24}$$

Denote

$$\begin{aligned}q_1(t, x) &= -\hat{w}_x(t, x)c(x)(\hat{z}^\alpha(t) - z^\alpha(t)),\\q_2(t, x) &= b_0\hat{w}(t)[L(t, x, \hat{z}^\alpha(t)) - L(t, x, z^\alpha(t))].\end{aligned}$$

Now (24) is rewritten as

$$\begin{aligned}0 =&\phi_t + \phi_{xx} + \phi_x[a(x) + c(x)z^\alpha(t)] - b_0\phi L(t, x, z^\alpha(t))\\&+ q_1(t, x) + q_2(t, x),\end{aligned} \tag{25}$$

where $\phi(T, x) = 0$.

We proceed to estimate the Hölder norm of q_1 and q_2. We have

$$|q_1(t, x)| \le |\hat{w}_x c|_0 \cdot |\hat{z}^\alpha - z^\alpha|_0, \quad \forall t, x. \tag{26}$$

Next we have (see e.g. [16])

$$[q_1]_{\gamma/2,\gamma} \le |\hat{w}_x c|_0 \cdot [\hat{z}^\alpha - z^\alpha]_{\gamma/2} + |\hat{z}^\alpha - z^\alpha|_0 \cdot [\hat{w}_x c]_{\gamma/2,\gamma}. \tag{27}$$

By (26) and (27), it follows that

$$|q_1|_{\gamma/2,\gamma} = |\hat{w}_x c(\hat{z}^\alpha - z^\alpha)|_{\gamma/2,\gamma} \le |\hat{w}_x c|_{\gamma/2,\gamma} \cdot |\hat{z}^\alpha - z^\alpha|_{\gamma/2}.$$

We continue to check q_2. Denote

$$\tilde{L}(t, x) = L(t, x, \hat{z}^\alpha(t)) - L(t, x, z^\alpha(t)).$$

Then we have

$$|\tilde{L}(t, x)| \le |L_z(t, x, \bar{z})| \cdot |\hat{z}^\alpha(t) - z^\alpha(t)|, \tag{28}$$

where $\bar{z}$ is some point between $\hat{z}^\alpha(t)$ and $z^\alpha(t)$. We have

$$|q_2(t, x)| \le b_0|\hat{w}(t)| \cdot |L_z(t, x, \bar{z})| \cdot |\hat{z}^\alpha(t) - z^\alpha(t)|,$$

so that

$$|q_2|_0 \le b_0|\hat{w}|_0 \cdot |\hat{z}^\alpha - z^\alpha|_0 \cdot \sup_{t,x,z} |L_z(t, x, z)|. \tag{29}$$

We further have

$$[q_2]_{\gamma/2,\gamma} \le b_0|\tilde{L}|_0 \cdot [\hat{w}]_{\gamma/2,\gamma} + b_0|\hat{w}|_0 \cdot |\tilde{L}|_{\gamma/2,\gamma}. \tag{30}$$

Next we estimate the Hölder norm of $\tilde{L}$. We have

$$|\tilde{L}(t_1, x_1) - \tilde{L}(t_2, x_2)| \le |\tilde{L}(t_1, x_1) - \tilde{L}(t_2, x_1) + \tilde{L}(t_2, x_1) - \tilde{L}(t_2, x_2)|.$$

By Corollary 2, for fixed x_1, we have

$$\frac{|\tilde{L}(t_1,x_1)-\tilde{L}(t_2,x_1)|}{|t_1-t_2|^{\gamma/2}} \le [\tilde{L}(\cdot,x_1)]_{\gamma/2} \le \widehat{C}_1|\hat{z}^\alpha - z^\alpha|_{\gamma/2}, \qquad \forall \hat{z}, z\in\mathcal{Z}. \tag{31}$$

The above estimate has used the bounds in (23) for $\hat{z}, z\in\mathcal{Z}$ and the local Lipschitz property of the Nemytskij operator $L(t,x,\cdot)$ acting on $z^\alpha(\cdot)\in C^{\gamma/2}([0,T])$. The constant $\widehat{C}_1$ depends only on $|\chi|_0, T, \sup_{t,x,z}(|L_t|+|L_z|+|L_{zt}|+|L_{zz}|)$ (see the selection of the parameters k_a, k_b, k' in Corollary 2). Next we have

$$\begin{aligned}\tilde{L}(t,x_1)-\tilde{L}(t,x_2) &= L(t,x_1,\hat{z}^\alpha(t)) - L(t,x_1,z^\alpha(t)) - L(t,x_2,\hat{z}^\alpha(t)) + L(t,x_2,z^\alpha(t))\\ &= \hat{L}(t,x_1,x_2,\hat{z}^\alpha(t)) - \hat{L}(t,x_1,x_2,z^\alpha(t))\\ &= \hat{L}_z(t,x_1,x_2,\bar{z})(\hat{z}^\alpha(t)-z^\alpha(t))\\ &= [L_z(t,x_1,\bar{z})-L_z(t,x_2,\bar{z})](\hat{z}^\alpha(t)-z^\alpha(t))\end{aligned}$$

where $\hat{L}(t,x_1,x_2,z) \coloneqq L(t,x_1,z)-L(t,x_2,z)$. Hence we have

$$\frac{|\tilde{L}(t,x_1)-\tilde{L}(t,x_2)|}{|x_1-x_2|^\gamma} \le \sup_{t,x,z}|L_{zx}(t,x,z)|^\gamma \cdot \sup_{t,x,z}(2|L_z(t,x,z|)^{1-\gamma}\cdot|\hat{z}^\alpha - z^\alpha|_0. \tag{32}$$

Subsequently, by (28), (31) and (32), for some constant $\widehat{C}_2$, we have

$$|\tilde{L}|_{\gamma/2,\gamma} \le \widehat{C}_2|\hat{z}^\alpha - z^\alpha|_{\gamma/2}. \tag{33}$$

Finally, by (29) and (33) we conclude

$$|q_2|_{\gamma/2,\gamma} \le \widehat{C}_3|\hat{z}^\alpha - z^\alpha|_{\gamma/2}, \qquad \forall z,\hat{z}\in\mathcal{Z}.$$

The constant $\widehat{C}_3$ depends only on $|\chi|_0, T, \gamma, \sup_{t,x,z}(|L_t|+|L_z|+|L_{zt}|+|L_{zx}|+|L_{zz}|)$. The lemma then follows from an application of Theorem 1. □

3.2 The FPK equation

For comparing two solutions, we take $\hat{z}\in\mathcal{Z}$ and introduce another equation

$$\begin{aligned}\hat{m}_t^\alpha(t,x) =& \hat{m}_{xx}^\alpha(t,x) - \hat{m}_x^\alpha[a(x)-2b_0\hat{V}_x^\alpha(t,x)+c(x)\hat{z}^\alpha(t)]\\ &- \hat{m}^\alpha\partial_x[a(x)-2b_0\hat{V}_x^\alpha(t,x)+c(x)\hat{z}^\alpha(t)],\end{aligned} \tag{34}$$

where $\hat{m}^\alpha(0,x)=p^\alpha(x)$. By Proposition 1, there is a unique solution $\hat{m}^\alpha$.

Lemma 2. *There exists a constant C_5^* such that for all $z,\hat{z}\in\mathcal{Z}$, we have*

$$|m^\alpha - \hat{m}^\alpha|_{1+\gamma/2,2+\gamma;Q_T} \le C_5^*|z^\alpha - \hat{z}^\alpha|_{\gamma/2;[0,T]}.$$

Proof. Denote $\phi = m^\alpha - \hat{m}^\alpha$. Then we have

$$\begin{aligned}\partial_t\phi(t,x) =& \partial_{xx}\phi - \partial_x\phi\cdot[a(x)-2b_0V_x^\alpha(t,x)+c(x)z^\alpha(t)]\\ &- \phi\cdot\partial_x[a(x)-2b_0V_x^\alpha(t,x)+c(x)z^\alpha(t)]\\ &+ \hat{m}_x\cdot[2b_0(\hat{V}_x^\alpha - V_x^\alpha)-c(x)(\hat{z}^\alpha(t)-z^\alpha(t))]\\ &+ \hat{m}\cdot\partial_x[2b_0(\hat{V}_x^\alpha - V_x^\alpha)-c(x)(\hat{z}^\alpha(t)-z^\alpha(t))],\end{aligned} \tag{35}$$

where $\phi(0,x)=0$. Denote

$$\kappa_1(t,x) = a(x)-2b_0V_x^\alpha + c(x)z^\alpha(t),$$

$$\kappa_2(t,x) = \partial_x[a(x) - 2b_0 V_x^\alpha(t,x) + c(x)z^\alpha(t)],$$
$$\kappa_3(t,x) = \hat{m}_x \cdot [2b_0(\hat{V}_x^\alpha - V_x^\alpha) - c(x)(\hat{z}^\alpha(t) - z^\alpha(t))],$$
$$\kappa_4(t,x) = \hat{m} \cdot \partial_x[2b_0(\hat{V}_x^\alpha - V_x^\alpha) - c(x)(\hat{z}^\alpha(t) - z^\alpha(t))].$$

We first have

$$[\kappa_1(t,x)]_{\gamma/2,\gamma} \le [a]_\gamma + 2b_0[V_x^\alpha]_{\gamma/2,\gamma} + [cz^\alpha]_{\gamma/2,\gamma}.$$

We further use the interpolation inequality in [16] to estimate $[V_x^\alpha]_{\gamma/2,\gamma}$ and get

$$|\kappa_1|_{\gamma/2,\gamma} \le |a|_\gamma + 2\widehat{C}_4 b_0 |V^\alpha|_{1+\gamma/2,2+\gamma} + |c|_\gamma |z^\alpha|_{\gamma/2}.$$

We note that the constant $\widehat{C}_4$ above depends on T but not on $(a(\cdot), b, c(\cdot), L(\cdot))$ in the model of the GMFG. It gets larger when T becomes smaller. We write

$$\kappa_2 = a_x(x) - 2b_0 V_{xx}^\alpha + c_x(x) z^\alpha(t).$$

Then

$$|\kappa_2(t,x)| \le |a_x(x)| + 2b_0|V_{xx}^\alpha| + |c_x(x)| \cdot |z^\alpha(t)|.$$

Next we have

$$[\kappa_2]_{\gamma/2,\gamma} \le [a_x]_\gamma + 2b_0[V_{xx}^\alpha]_{\gamma/2,\gamma} + [c_x z^\alpha]_{\gamma/2,\gamma}.$$

Now it follows that

$$|\kappa_2|_{\gamma/2,\gamma} \le |a|_{1+\gamma} + 2b_0|V^\alpha|_{1+\gamma/2,2+\gamma} + |c|_{1+\gamma} \cdot |z^\alpha|_{\gamma/2}.$$

We have

$$|\kappa_3(t,x)| \le |\hat{m}_x(t,x)| \cdot (2b_0|V_x^\alpha - \hat{V}_x^\alpha| + |c(x)| \cdot |\hat{z}^\alpha(t) - z^\alpha(t)|),$$

where we use the relation $V^\alpha = -\ln w^\alpha / b_0$ to get

$$\begin{aligned} |V_x^\alpha - \hat{V}_x^\alpha| &\le \widehat{C}_5(|w(t,x) - \hat{w}(t,x)| + |w_x - \hat{w}_x|) \\ &\le \widehat{C}_5 C_4^* |z^\alpha - \hat{z}^\alpha|_{\gamma/2} \qquad \forall t \in [0,T],\ x \in \mathbb{R}, \end{aligned}$$

by Lemma 1. Next we have

$$\begin{aligned} [\kappa_3]_{\gamma/2,\gamma} \le & |\hat{m}_x|_{0;Q_T} \cdot [2b_0(V_x^\alpha - \hat{V}_x^\alpha) + c(\hat{z}^\alpha - z^\alpha)]_{\gamma/2,\gamma;Q_T} \\ & + [\hat{m}_x]_{\gamma/2,\gamma;Q_T} |2b_0(V_x^\alpha - \hat{V}_x^\alpha) + c(\hat{z}^\alpha - z^\alpha)|_{0;Q_T}, \end{aligned}$$

where we estimate

$$|V_x^\alpha - \hat{V}_x^\alpha|_{\gamma/2,\gamma} \le \widehat{C}_6(|w - \hat{w}|_{\gamma/2,\gamma} + |w_x - \hat{w}_x|_{\gamma/2,\gamma}) \tag{36}$$

using Lemma 3 by writing $V_x = G(w, w_x)$. The form of G can be easily determined. The constant $\widehat{C}_6$ is determined only using the upper bound C_w for $|w|_{\gamma/2,\gamma}$ and $\sup_{|x|,|y| \le C_w}(|G_x| + |G_y| + |G_{xx}| + |G_{yy}| + |G_{xy}|)$ for $G(x,y)$. By the interpolation inequality [16] and Lemma 1, we find an upper bound for the right hand side of (36) in terms of $|\hat{z}^\alpha - z^\alpha|_{\gamma/2}$, which further leads to the estimate

$$|\kappa_3|_{\gamma/2,\gamma} \le \widehat{C}_7 |\hat{z}^\alpha - z^\alpha|_{\gamma/2}.$$

Finally, by writing V_{xx}^α in terms of (w, w_x, w_{xx}) and using Lemma 3 and the interpolation inequality, we similarly get

$$|\kappa_4|_{\gamma/2,\gamma} \le \widehat{C}_8 |\hat{z}^\alpha - z^\alpha|_{\gamma/2},$$

for some constant $\widehat{C}_8$. The lemma follows from applying Theorem 1 to (35). □

4 Perturbation estimate of the graphon coupling function

Throughout this section we suppose that Assumptions (A1), (A2), (A3), (A4) and (A5) hold.

Given $z \in \mathcal{Z}$, we use Theorem 2 and Proposition 1 to determine (V^α, m^α) in (5) and (7) for each $\alpha \in [0,1]$. Define the new graphon coupling function

$$z_1^\alpha(t) = (\Phi z)^\alpha(t) := \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x) m^\beta(t,x) dx d\beta,$$

Since $m^\beta \in C^{1+\gamma/2, 2+\gamma}([0,T]\times\mathbb{R})$, $z_1^\alpha(t)$ is Hölder continuous in t. For $\hat{z} \in \mathcal{Z}$, we similarly obtain $\hat{m}^\alpha$, $\beta \in [0,1]$ and denote

$$\hat{z}_1^\alpha(t) = \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)\chi(x)\hat{m}^\beta(t,x) dx d\beta,$$

By the interpolation inequality [16], there exists a constant $\widehat{C}_9$ (which depends on (T,γ)) such that for each $f \in C^{\gamma/2,\gamma}([0,T]\times\mathbb{R})$ we have

$$[f]_{\gamma,\gamma/2;Q_T} \le \widehat{C}_9 |f|_{1+\gamma/2,2+\gamma;Q_T}, \qquad Q_T = [0,T]\times\mathbb{R}. \tag{37}$$

We state the key result on Lipschitz continuity of the operator Φ.

Theorem 3. *For all $z, \hat{z} \in \mathcal{Z}$, we have*

$$\sup_\alpha |z_1^\alpha - \hat{z}_1^\alpha|_{\gamma/2;[0,T]} \le (\widehat{C}_9 + 1) C_g C_\chi C_5^* \sup_\alpha |z^\alpha - \hat{z}^\alpha|_{\gamma/2;[0,T]},$$

where $C_g = \sup_\alpha \int_0^1 |g(\alpha,\beta)| d\beta$ and $C_\chi = \int_{\mathbb{R}} |\chi(x)| dx$.

Proof. Denote

$$\tilde{z}_1^\alpha = z_1^\alpha - \hat{z}_1^\alpha, \qquad \tilde{m}^\alpha = m^\alpha - \hat{m}^\alpha.$$

We have

$$\begin{aligned} \frac{|\tilde{z}_1^\alpha(t) - \tilde{z}_1^\alpha(s)|}{|t-s|^{\gamma/2}} &\le \int_0^1 \int_{\mathbb{R}} g(\alpha,\beta)|\chi(x)| \frac{|\tilde{m}^\beta(t,x) - \tilde{m}^\beta(s,x)|}{|t-s|^{\gamma/2}} dx d\beta \\ &\le C_g \int_{\mathbb{R}} |\chi(x)| dx \cdot \sup_\beta [\tilde{m}^\beta]_{\gamma,\gamma/2} \\ &= C_g C_\chi \sup_\beta [\tilde{m}^\beta]_{\gamma,\gamma/2}. \end{aligned} \tag{38}$$

On the other hand, it is easily seen that

$$|z_1^\alpha - \hat{z}_1^\alpha|_{0;[0,T]} \le C_g C_\chi \sup_\beta |m^\beta - \hat{m}^\beta|_{0;Q_T}.$$

Now applying inequality (37) to $\tilde{m}$, we have

$$[\tilde{m}^\alpha]_{\gamma,\gamma/2} \le \widehat{C}_9 |\tilde{m}^\alpha|_{1+\gamma/2,2+\gamma;Q_T}, \quad \forall \alpha \in [0,1].$$

We conclude that

$$\sup_\alpha |z_1^\alpha - \hat{z}_1^\alpha|_{\gamma/2;[0,T]} \le (\widehat{C}_9 + 1) C_g C_\chi \sup_\alpha |m^\alpha - \hat{m}^\alpha|_{1+\gamma/2,2+\gamma;Q_T}. \tag{39}$$

By Lemma 2 and (39), we have

$$\sup_\alpha |z_1^\alpha - \hat{z}_1^\alpha|_{\gamma/2;[0,T]} \le (\widehat{C}_9 + 1) C_g C_\chi C_5^* \sup_\alpha |z^\alpha - \hat{z}^\alpha|_{\gamma/2;[0,T]},$$

which gives the required Lipschitz property of the operator Φ. □

If the coefficient $(\widehat{C}_9+1)C_g C_\chi C_5^*$ is less than 1, we obtain a contraction, which implies the existence and uniqueness of a solution to the GMFG equation system.

Corollary 1. *If* $(\widehat{C}_9 + 1)C_g C_\chi C_5^* < 1$*, the GMFG equation system* (8)–(10) *has a unique solution* $(V^\alpha, m^\alpha)_{0\le\alpha\le 1}$ *with* $V^\alpha, m^\alpha \in C^{1+\gamma/2, 2+\gamma}([0,T]\times\mathbb{R})$. □

Remark 5. If χ is only bounded without the integrability property, the above estimate in (38) is not valid.

Remark 6. If either $|c|_{1+\gamma} + b_0$ (where $|c|_{1+\gamma} \coloneqq |c|_0 + |c_x|_0 + [c_x]_\gamma$) or $C_g C_\chi$ is sufficiently small, the contraction condition in Corollary 1 is satisfied.

We illustrate how to construct a concrete model to verify the contraction condition in Corollary 1. We start with any reference model (M_{ref}) consisting of

$$(a(x), b, c(x), g, \chi(x), L(\cdot), r, T)$$

and determine the parameters $(|\chi|_0, C_3^*)$ in (23). Now we specify $\mathcal{Z}$ with the two fixed parameters $(|\chi|_0, C_3^*)$. We next construct a new model (M_{new}) by replacing $(c(x), r)$ by $(\epsilon c(x), r/\epsilon)$ with a small positive number ϵ while all other entries in model (M_{ref}) remain unchanged. We still use the same set $\mathcal{Z}$ in model (M_{new}) for which we can make C_4^* and C_5^* sufficiently close to zero if ϵ is sufficiently close to zero. Thus the new model can verify the contraction condition with a small ϵ, indicating weak dynamical coupling and expensive control.

Appendix

For the reader's convenience, we provide some standard materials on the Nemytskij operator. The reader may see more systematic development of the subject in [1, 10, 14]. Consider two Hölder spaces $H_n^\gamma \coloneqq C^\gamma([a,b];\mathbb{R}^n)$ and $H_1^\gamma \coloneqq C^\gamma([a,b];\mathbb{R})$, where $\gamma \in (0,1)$. For a function $f : [a,b]\times\mathbb{R}^n \to \mathbb{R}$, its Nemytskij operator is defined by

$$(\mathbf{F}h)(t) = f(t, h(t)), \qquad h \in H_n^\gamma.$$

We summarize results on $\mathbf{F}$ as a mapping between Hölder spaces. Following [14], we introduce the following conditions for a function ϕ:

Condition (A) – For each compact set $S \subset \mathbb{R}^n$, there exists a constant $k_A \coloneqq k_A(\phi, S)$ such that

$$|\phi(t,y) - \phi(s,y)| \le k_A|t-s|^\gamma, \quad \forall t, s \in [a,b], \qquad \forall y \in S. \tag{A.1}$$

Condition (B) – For each compact set $S \subset \mathbb{R}^n$, there exists a constant $k_B \coloneqq k_B(\phi, S)$ such that

$$|\phi(t,y) - \phi(s,z)| \le k_B(|t-s|^\gamma + |y-z|), \qquad \forall t, s \in [a,b], \quad \forall y, z \in S. \tag{A.2}$$

Clearly, condition (B) for ϕ implies condition (A) for ϕ.

Fix any constant $k_0 > 0$. Let $\bar{B}(0,k_0) \subset \mathbb{R}^n$ be the closed ball centered at the origin with radius k_0, and define

$$C_{f,k_0} = \max_{a\le t\le b, |x|\le k_0} |f_x(t,x)|,$$

$$M_{f,k_0} = (\sum_{i=1}^n l_{f,k_0,i}^2)^{1/2}, \qquad l_{f,k_0,i} = k_B(f_{x_i}, \bar{B}(0,k_0)).$$

Theorem A.1. *[14, Theorem 3] Suppose that* $f(t,x)$ *satisfies condition (A), and each partial derivative* $f_{x_i}(t,x)$ *satisfies condition (B). Then the operator* $\mathbf{F}$ *is from* H_n^γ *to* H_1^γ *and is locally Lipschitz continuous, i.e., for any given constant* k_0 *and any* h_1, h_2 *with* $\|h_k\|_{H_0^\gamma} \le k_0$*, we have*

$$\|\mathbf{F}h_1 - \mathbf{F}h_2\|_{H_1^\gamma} \le C_0^f \|h_1 - h_2\|_{H_n^\gamma},$$

where

$$C_0^f = C_{f,k_0} + M_{f,k_0}(1 + 2k_0).$$

Ref. [14] proved the local Lipschitz property of the operator $\mathbf{F}$. The constant C_0^f does not use k_A in condition (A), which is only used to show that $\mathbf{F}$ maps $h \in H_n^\gamma$ to the Hölder space H_1^γ. The constant C_0^f here is determined by keeping track of the estimates in [14]. Specifically, we have

$$|\mathbf{F}h_1(t) - \mathbf{F}h_2(t)| \le C_{f,k_0} \sup_t |h_2(t) - h_1(t)|. \tag{A.3}$$

Let $d(t) = \mathbf{F}(h_1)(t) - \mathbf{F}(h_2)(t)$. Let $[h]_\gamma$ denote the Hölder semi-norm of $h \in H_n^\gamma$. Then by the method in [14, p. 114],

$$\begin{aligned} |d(t) - d(s)| \cdot |t-s|^{-\gamma} \le & C_{f,k_0}[h_2 - h_1]_\gamma \\ & + M_{f,k_0}(1 + [h_1]_\gamma + [h_2]_\gamma) \sup_t |h_2(t) - h_1(t)|. \end{aligned} \tag{A.4}$$

By (A.3) and (A.4), we have

$$\begin{aligned} \|\mathbf{F}(h_1) - \mathbf{F}(h_2)\|_{H_1^\gamma} &\le C_{f,k_0}\|h_2 - h_1\|_{H_n^\gamma} + M_{f,k_0}(1+2k_0) \sup_t |h_2(t) - h_1(t)| \\ &\le [C_{f,k_0} + M_{f,k_0}(1+2k_0)] \cdot \|h_1 - h_2\|_{H_n^\gamma} \end{aligned}$$

for all h_1, h_2 satisfying $\|h_i\|_{H_n^\gamma} \le k_0$. The last inequality shows the choice of C_0^f in Theorem A.1.

A.1 Application to the graphon model

Now consider the function $L(t, x, z)$ defined on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Suppose that for some constants k_a, k_b, k', there hold the inequalities

$$\begin{aligned} |L(t,x,z) - L(s,x,z)| &\le k_a |t-s|^{\gamma/2}, \\ |L_z(t,x,z_1) - L_z(s,x,z_2)| &\le k_b(|t-s|^{\gamma/2} + |z_1 - z_2|), \\ |L_z(t,x,z)| &\le k', \qquad \forall t \in [0,T], x, z, z_1, z_2. \end{aligned}$$

Denote $\mathbf{F}h(t) = L(t, x, h(t))$ for $h \in C^{\gamma/2}([0,T])$, where x is regarded as a fixed value. The Hölder norm (resp., semi-norm) of h is simply written as $|h|_{\gamma/2}$ (resp., $[h]_{\gamma/2}$).

Corollary 2. *Suppose $h_1, h_2 \in C^{\gamma/2}([0,T])$, and $|h_i|_{\gamma/2} \le k_0$. Then*

$$|\mathbf{F}(h_1) - \mathbf{F}(h_2)|_{\gamma/2} \le [k' + k_b(1+2k_0)] \cdot |h_1 - h_2|_\gamma.$$

Proof. In analogue to (A.3), for fixed x, we have

$$|\mathbf{F}h_1(t) - \mathbf{F}h_2(t)| \le k' \sup_t |h_2(t) - h_1(t)|.$$

Let $d(t) = \mathbf{F}(h_1)(t) - \mathbf{F}(h_2)(t)$. By the method in (A.4), we have

$$\begin{aligned} |d(t) - d(s)| \cdot |t-s|^{-\gamma/2} \le & k'[h_2 - h_1]_{\gamma/2} \\ & + k_b(1 + [h_1]_{\gamma/2} + [h_2]_{\gamma/2}) \sup_t |h_2(t) - h_1(t)|. \end{aligned}$$

Hence for fixed x, we have

$$\begin{aligned} |\mathbf{F}(h_1) - \mathbf{F}(h_2)|_{\gamma/2} &\le k'|h_2 - h_1|_{\gamma/2} + k_b(1+2k_0) \sup_t |h_2(t) - h_1(t)| \\ &\le [k' + k_b(1+2k_0)] \cdot |h_1 - h_2|_{\gamma/2} \end{aligned}$$

for all h_1, h_2 satisfying $|h_i|_{\gamma/2} \le k_0$.

A.2 Operator acting on functions of time and space

In the following we make an extension to vector-valued Hölder continuous functions v defined on $[0,T]\times\mathbb{R}^n$. Let $G:\mathbb{R}^k\to\mathbb{R}$ be a function with continuous partial derivatives $G_{\xi_i}(\xi)$, and $G_{\xi_i\xi_j}(\xi)$ for $1\le i,j\le k$. Denote $H^{\gamma/2,\gamma}=C^{\gamma/2,\gamma}([0,T]\times\mathbb{R}^n;\mathbb{R}^k)$ with $\gamma\in(0,1)$.

Denote the operator

$$(\mathbf{G}v)(t,x)=G(v(t,x)), \qquad\qquad v\in H^{\gamma/2,\gamma}.$$

Lemma 3. *The operator* $\mathbf{G}$ *maps* $H^{\gamma/2,\gamma}$ *to* $C^{\gamma/2,\gamma}([0,T]\times\mathbb{R}^n;\mathbb{R})$*, and is locally Lipschitz continuous.*

Proof. Take any positive constants C_1 and C_2. The Hölder norm of $h\in H^{\gamma/2,\gamma}$ will be simply written as $|h|_{\gamma/2,\gamma}$. Denote the set

$$\mathcal{H}_{C_1,C_2}=\{v\in H^{\gamma/2,\gamma}:|v|_0\le C_1,|v|_{\gamma/2,\gamma}\le C_2\}.$$

Take $v,\hat{v}\in\mathcal{H}_{C_1,C_2}$. We have

$$|G(v(t,x))-G(v(s,y))|\le l_1|v(t,x)-v(s,y)|.$$

where $l_1=\max_{|\xi|\le C_1}|G_\xi(\xi)|$. Next, we use the Hölder seminorm of v to get

$$|G(v(t,x))-G(v(s,y))|\le l_1[v]_{\gamma/2,\gamma}(|t-s|^{1/2}+|x-y|)^\gamma.$$

It follows that

$$|\mathbf{G}v|_{\gamma/2,\gamma}\le\max_{|\xi|\le C_1}|G(\xi)|+l_1[v]_{\gamma/2,\gamma}. \tag{A.5}$$

So $\mathbf{G}$ is a mapping from $H^{\gamma/2,\gamma}$ to $C^{\gamma/2,\gamma}([0,T]\times\mathbb{R}^n;\mathbb{R})$.

We proceed to estimate the Hölder norm of $v_1\coloneqq\mathbf{G}v-\mathbf{G}\hat{v}$. We have

$$|v_1|_0\le l_1|v-\hat{v}|_0.$$

We further write

$$\begin{aligned}v_1(t,x)&=(v(t,x)-\hat{v}(t,x))\int_0^1 G_y(\hat{v}(t,x)+\tau[v(t,x)-\hat{v}(t,x)])d\tau,\\ v_1(s,y)&=(v(s,y)-\hat{v}(s,y))\int_0^1 G_y(\hat{v}(s,y)+\tau[v(s,y)-\hat{v}(s,y)])d\tau.\end{aligned}$$

Now we have

$$\begin{aligned}&v_1(t,x)-v_1(s,y)\\ &=[v(t,x)-\hat{v}(t,x)-(v(s,y)-\hat{v}(s,y))]\int_0^1 G_y(\hat{v}(t,x)+\tau[v(t,x)-\hat{v}(t,x)])d\tau\\ &\quad+(v(s,y)-\hat{v}(s,y))\int_0^1\{G_y(\hat{v}(t,x)+\tau[v(t,x)-\hat{v}(t,x)])\\ &\quad-G_y(\hat{v}(s,y)+\tau[v(s,y)-\hat{v}(s,y)])\}d\tau\end{aligned}$$

Denote $\lambda_{G,1}=\max_{|\xi|\le C_1}(\sum_{i,j}|G_{\xi_i\xi_j}(\xi)|^2)^{1/2}$. Hence we have

$$\begin{aligned}|v_1(t,x)-v_1(s,y)|\le{}&[v-\hat{v}]_{\gamma/2,\gamma}l_1(|t-s|^{1/2}+|x-y|)^\gamma\\ &+|v(s,y)-\hat{v}(s,y)|\lambda_{G,1}\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^1 [\tau|v(t,x) - v(s,y)| + (1-\tau)|\hat{v}(t,x) - \hat{v}(s,y)|]d\tau \\
&\leq [v - \hat{v}]_{\gamma/2,\gamma} l_1 (|t-s|^{1/2} + |x-y|)^\gamma \\
&\quad + |v - \hat{v}|_0 \lambda_{G,1} C_2 (|t-s|^{1/2} + |x-y|)^\gamma.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|\mathbf{G}v - \mathbf{G}\hat{v}|_{\gamma/2,\gamma} &\leq (l_1 + \lambda_{G,1} C_2)|v - \hat{v}|_0 + l_1 [v - \hat{v}]_{\gamma/2,\gamma} \\
&\leq (l_1 + \lambda_{G,1} C_2)|v - \hat{v}|_{\gamma/2,\gamma}.
\end{aligned}
$$

This completes the proof. □

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