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Sparse network mean field games: Ring structures and related topologiess

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Abstract : For sequences of networks embedded in the unit cube $[0,1]^m$, (weak) measure limits of sequences of empirical measures of vertex densities (vertexon functions) exist, and the associated (weak) measure limits of sequences of empirical measures of edge densities (graphexon functions) in $[0,1]^{2m}$ exist, regardless of the sparsity or density of the limit graphs. This paper presents an extension of Graphon Mean Field Game (GMFG) theory to the vertexon-graphexon MFG set-up (denoted GXMFG). Specific second order dynamics are introduced for the inter-node influence mediated by the singular part of a network graphexon measure; this is analyzed in the particular cases of a network limit ring topology and a limit rectangular lattice topology. Existence and uniqueness results are presented for the corresponding GXMFG equations.

Keywords : Mean field games, graphon, graphexon, Nash equilibria, decentralized control

Résumé : Pour les séquences de réseaux plongés dans le cube unité $[0,1]^m$, il existe des limites de mesure (faibles) de séquences de mesures empiriques de densités de sommets (nomées fonctions de vertexons), et les limites de mesure (faibles) associées de séquences de mesures empiriques de densités d'arêtes (nomées fonctions de graphexons) dans $[0,1]^{2m}$ existent, indépendamment de la rareté ou de la densité des graphes limites. Cet article présente une extension de la théorie du jeu de champ moyen du graphon (GMFG) à la configuration MFG vertexon-graphexon (notée GXMFG). Une dynamique spécifique du second ordre est introduite pour l'influence inter-nœuds médiatisée par la partie singulière d'une mesure de graphexon de réseau; ceci est analysé dans les cas particuliers d'une topologie en anneau limite de réseau et d'une topologie en treillis rectangulaire limite. Les résultats d'existence et d'unicité sont présentés pour les équations GXMFG correspondantes.

Mots clés : Jeux à champ moyen, graphon, graphexon, équilibres de Nash, contrôle décentralisé

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1 Introduction

1.1 GMFG and GXMFG theory

In [8], the basic existence and uniqueness results were established for the Graphon Mean Field Game equations, and a full derivation of the fundamental results was provided, including the key ϵ-Nash theory for GMFG systems which relates the infinite population equilibria on infinite networks to finite population equilibria on finite networks [6, 7]. The GMFG equations are of great generality since they permit the study, in the limit, of dense infinite networks of non-cooperative dynamical agents concentrated in asymptotically infinite sub-populations on the asymptotically infinite set of nodes. Moreover the classical MFG equations are retrieved when communication over the infinite network is restricted to uniform weightings of direct influences of all agents on the network on every other agent in the network.

In common with the GMFG scheme, the Graphexon MFG (GXMFG) system of equations [9] is given by the linked equations on $[0, T]$ for (i) the Hamilton-Jacobi-Bellman (HJB) PDE for the value function V_α for a generic agent's stochastic control problem at node α, (ii) the Fokker-Planck-Kolmogorov (FPK) equation for the McKean-Vlasov stochastic differential equation (SDE) for the local mean field μ_α of the generic agent, and (iii) the specification of the best response feedback law.

The GMFG framework is formulated in terms of graphons and hence is restricted to asymptotically dense networks, whereas the GXMFG construction is formulated within the graphexon setting and applies to networks which are asymptotically a combination of sparse and dense graph limits.

The GXMFG formalism in this paper incorporates a second order differential interaction term in the dynamics. This interaction structure is used to model the influence of the neighboring subpopulations through the spatial variations of their behaviors. This extends the first order interaction introduced in previous work [9].

1.2 Graphons, vertexons and graphexons

Graphon theory ([23]) provides a natural framework for the formulation of game theoretic problems involving agents distributed over large networks when the nodes can be indexed by the reals in $[0, 1]$ furnished only with the topology of the node set $[0, 1]$. Graphons are then measurable bounded functions on $[0, 1]^2$ (interpreted as generalized adjacency matrices), and the topology on the space of graphons is that of the cut metric (see [23, Chapter 8]). This is the graphon framework which is commonly used in dynamics and games analysis on large networks (see e.g. [1–3, 8, 10, 12, 13, 18, 24–26]).

An (embedded) vertexon in a connected compact set M in $\mathbb{R}^m$ is defined to be the vertex set of a graph together with its empirical densities (and the associated measures) generated by an asymptotically dense partition hierarchy of M. It is shown in [4] that sequences of vertexons have subsequential vertexon limit measures in M. Consequently, the differentiation of functions on vertexon limits with open support is well defined, which is not the case for graphons where no metric topology is defined. Moreover, along such sequences, the associated sequences of graphs have subsequential graph edge limit measures, termed graphexons and, unlike the graphon case, non-empty graphexon limits are defined for all non-empty graph sequences, including sparse graph sequences. Such large sparse graphs occur, for instance, in biological neural networks and electrical power grids. [15–17, 19].

1.3 Organization

The paper is organized as follows. Section 2 provides a concise overview of the notion of vertexon-graphexon measure limits. Section 3 introduces the second order mean field dynamics model for agent interactions on certain singular graphexon network limits. The graphexon mean field game equations on the ring are presented in Section 4. The linear quadratic GXMFG problem on the ring is developed in Section 5, and its extension to the infinite rectangular lattice is given in Section 6. Finally, the corresponding existence and uniqueness results for the GXMFG equations are established together with a numerical example in Section 7.

2 Vertexon-graphexon limits of embedded graph sequences

Let $\{G_n = (V_n, E_n),\ n \in \mathbb{N}\}$ be a sequence of (not necessarily increasing) finite non-empty simple graphs, that is to say a sequence of vertex set – edge set pairs, where each finite non-empty graph G_n has no more than one undirected edge $(v_i, v_j) \in E_n \subset V_n \times V_n$ between any two nodes $v_i, v_j \in V_n$ and no node v_i has a self-loop (v_i, v_i). The set of all infinite sequences of vertex sets shall be denoted $\mathcal{V}$.

It will be assumed that $\{G_n, n \in \mathbb{N}\}$ is an *embedded graph* sequence, that is to say $V_n \subset M, n \in \mathbb{N}$ where $M := [0,1]^m \subset \mathbb{R}^m$, and that each V_n, termed a *vertexon*, is non-empty. Henceforth, $\{G_n = (V_n, E_n),\ n \in \mathbb{N}\}$ denotes a sequence of graphs with nodes in M, and edges (taken as line segments) lie in M, but, as a set, $E_n \subset V_n \times V_n$.

Definition 2.1. Let $\mathcal{P} := \{P_k, k \in \mathbb{N}\}$ denote a sequence of *lattice equipartitions of M, where each* $P_k, k \in \mathbb{N}$, *consists of the set of disjoint voxel sets*

$$\begin{aligned} &P_{k,\underline{i}} = \prod\nolimits_{t=1}^{m} \tilde{(}\frac{i_t - 1}{2^{k+1}}, \frac{i_t}{2^{k+1}}] \subset M, \\ &\underline{i} = (i_1, .., i_t, .., i_m), \\ &i_t \in [2^{k+1}] = \{1, ..., 2^{k+1}\},\ t \in [m] := \{1, ..., m\}, \end{aligned} \tag{1}$$

where $\prod_{t=1}^{m}$ *denotes Cartesian product,* $\tilde{(}\frac{i_t-1}{2^{k+1}}, \frac{i_t}{2^{k+1}}]$ *denotes* $(\frac{i_t-1}{2^{k+1}}, \frac{i_t}{2^{k+1}}]$ *if* $i_t > 1$ *and denotes* $[\frac{i_t-1}{2^{k+1}}, \frac{i_t}{2^{k+1}}]$ *in case* $i_t = 1$.

Definition 2.2 (Vertexon (Voxel Stepping) Functions ([4])). For fixed $m \in \mathbb{N}$, and any P, a *vertexon function*, $U_{n,k,\underline{i}}(z), z \in P_{k,\underline{i}}$, of a vertexon $V_n, n \in \mathbb{N}$, is defined by setting

$$\begin{aligned} &U_{n,k,\underline{i}}(z) = \frac{1}{card(V_n)} \frac{card(V_n \cap P_{k,\underline{i}})}{\mu(\{P_{k,\underline{i}})\}}, \\ &z \in P_{k,\underline{i}},\ \underline{i} = (i_1, ..., i_m),\ i_t \in [2^{k+1}],\ t \in [m], \end{aligned} \tag{2}$$

where $\mu(A)$ denotes the Lebesgue measure of a Borel set A.

Evidently, $U_{n,k,\underline{i}}(z), z \in P_{k,\underline{i}}$, equals the fraction of the vertices of V_n lying in $P_{k,\underline{i}}$ per unit volume of $[0,1]^m$. Furthermore, the value of $U_{k,i}(z)$ is independent of the value of $z \in P_{k,\underline{i}}$ and so $U_{k,i}(\cdot)$ is constant over $P_{k,\underline{i}}$ for each $\underline{i}$.

Theorem 2.3 (Vertexon (Stepping Function) Limits [4]). Let $V^{[\infty]} := \{V_n, n \in \mathbb{N}\}$ be a sequence of vertexons. Then, any sequence $\{U_{n,k}(z); z \in M, n, k \in \mathbb{N}\}$ of vertexon functions derived from $V^{[\infty]}$ possesses a subsequence $\{U_{n_{k,k,}}(z); z \in M\}$ converging weakly in $L_1(M)$ to a limit *vertexon (measure)* $V_\infty(dz)$, $z \in M$.

Definition 2.4 (Edge (Voxel Stepping) Functions [4])). For fixed $m \in \mathbb{N}$, an *edge voxel stepping function,*, $E^n_k, \underline{i}, \underline{j}(z), z \in P_{k,\underline{i}} \times P_{k,\underline{j}} \in P^2_k, P_k \in \mathcal{P}$, for the edge set E_n of a graph $G_n = (V_n, E_n)$, with $Card(V_n) \geq 2), Card(E_n) \geq 1$, is defined as

$$\begin{aligned} &E^n_{k,\underline{i},\underline{j}}(z) = \frac{1}{card(E_n)} \frac{1}{\mu(P_{k,\underline{i}})\mu(P_{k,\underline{j}})} \left[\sum\nolimits_{s,t} e^w_{s,t} : v_s \in V_n \cap P_{k,\underline{i}},\ v_t \in V_n \cap P_{k,\underline{j}}, e_{s,t} \in E^n \right], \\ &z \in P^2_{k,\underline{i}\underline{j}} := P_{k,\underline{i}} \times P_{k,\underline{j}}, \underline{i} = (i_1, ., i_m), \underline{j} = (j_1, ., j_m) \subset M,\ i_s, j_t \in [2^{k+1}],\ s, t \in [m]. \end{aligned}$$

We shall refer to vertexon and edge function sequences and their limits as *vertexon* and *graphexon* sequences, and on occasion, for simplicity, refer to the vertexon and graphexon sequence limits themselves as vertexons and graphexons.

Definition 2.5. ([4]) A *vertexon-graphexon measure* pair $V(dx) \in L_1(M^1), G(dz) \in L_1(M^2)$, is a pair of positive measures on the $m+2m$ dimensional unit cube $M^1 \times M^2 = [0,1]^m \times [0,1]^{2m}$.

Theorem 2.6 (Vertexon-Graphexon Limits (after [4])). Let $\{G_n = (V_n, E_n),\ n \in \mathbb{N}\}$ be an embedded graph sequence in M, together with the associated vertexon-graphexon function sequence pairs $\{U_n(x); E_n(z); n \in \mathbb{N}\}$. Then, there exists a sub-sequence $\{U_{n_p}(x); E_{n_p}(z); n_p, p \in \mathbb{N}; (x,z) \in M^1 \times M^2\}$ which converges weakly to the limit measure pair $V_\infty(dx), W_\infty(dz),\ x \in M,\ z \in M^2$.

In other words, vertexon-graphexon sequences of empirical measures converge weakly to limiting vertexon-graphexon measure pairs as in $G_{n_p} = (U_{n_p}, E_{n_p}) \quad \to \quad (V_\infty, W_\infty)$ as $n_p \to \infty$.

Example 2.7. *(Immediate Neighbour Connections on* $[0,1]$*)* Let $\{G_n = (V_n, E_n),\ n \in \mathbb{N}\}$ be such that V_n consists of n nodes uniformly distributed along the unit interval $[0,1]$ and E_n consists of the two edges of each node to its nearest neighbours in V_n. In this case, the unique limit vertexon $V_\infty(dz), z \in M$, is the unit density corresponding to the absolutely continuous unit measure on $[0,1]$, while the limit graphexon W_∞ is the unit measure $G(x,dy) = \delta_x$ concentrated on the diagonal $\{(x,x)|\ x \in [0,1]\}$ in the unit square $[0,1]^2$. This example may be specialized to the case where the end points indexed by $\{0\}$ and $\{1\}$ are mutually identified.

Example 2.8. [4] *(Erdös-Rényi Graphexon Limits in* $[0,1]^4$*)* Consider the iterative random construction of Erdös-Rényi type with vertices placed at random in $[0,1]^2$ and with edges assigned with probability $p, 0 < p < 1$. Here the limiting edge measure for the sequence of graphexons possesses a constant density function of value $p,\ 0 \le p \le 1$, on the 4-cube $[0,1]^4$.

Example 2.9. *(Vertexon-Graphexon Ring Network)* Consider a network of nodes uniformly distributed in the infinite limit on a circle of radius $\frac{1}{2}$ situated in $[0,1]^2$. This graph limit is given by a vertexon measure uniformly distributed on $\{(1/2 + (1/2)\cos\theta, 1/2 + (1/2)\sin\theta); \theta \in [0, 2\pi)\}$, where 2π is identified with 0.

Then it may be verified that the corresponding graphexon is given by the unit mass supported by the circle, or ring, $\{(1/2 + 1/2\cos\theta, 1/2 + 1/2\sin\theta, 1/2 + 1/2\cos\theta, 1/2 + 1/2\sin\theta]; \theta \in [0, 2\pi)\}$ lying in the four dimensional cube $[0,1]^4$; it is a one dimensional compact Riemannian manifold without boundary where 2π is identified with 0.

Example 2.10. *(Vertexon-Graphexon of the Infinite Rectangular Lattice)* Consider the limit of a uniformly distributed (uniformly) rectangular (i.e. square) grid in $[0,1]^2$. This gives rise to the vertexon given by the uniform unit density on $[0,1]^2$, which is evidently not singular.

Then the corresponding graphexon is given by the sum of two measures supported on two foliations of $\{[0,1]^4\}$ namely

$$\Big\{\frac{1}{2}\delta(p-x)\delta(q-y) + \frac{1}{2}\delta(p-y)\delta(q-x);\ 0 \le x, y, p, q \le 1\Big\}.$$

3 The ring topology: Second order interaction of immediate neighbors

In this section we model homeomorphically the one dimensional ring graphexon of Example 3 with the one dimensional line segment in the second case in Example 1 where the end points are identified with each other, which here are indexed by $\{0\}$ and $\{2\pi\}$.

In previous work [9], the graphon framework of [8] was extended to include the graphexon case of singular measures. In this paper, we first consider the limit of finite populations of stochastic dynamical systems interacting over finite networks with the ring graphexon limit. Specifically we adopt the following stochastic differential equation (SDE) model for the dynamics of a generic agent at a node α in the ring limit network:

$$\begin{aligned} dx_\alpha(t) =& f_0[x_\alpha(t), u_\alpha(t), \mu_\alpha(t)]dt \\ &+ \int_\beta \int_z g(\alpha,\beta) f_1(x_\alpha, u_\alpha, z)\mu_\beta(dz)d\beta dt \\ &+ \int_\beta f_2(x_\alpha, u_\alpha, \partial_\beta \langle \mu_\beta, \psi\rangle) N(\alpha, d\beta)dt \\ &+ \sigma dw_\alpha(t) \\ &=: (f_0 + \widetilde{f}_1 + \widetilde{f}_2)dt + \sigma dw_\alpha(t). \end{aligned} \tag{3}$$

Here f_1 describes the coupling with subpopulations at other locations of the dense subnetwork expressed via the absolutely continuous part of the graphexon measure given by the graphon function g, while f_2 describes the coupling with subpopulations at other locations of the sparse subnetwork expressed via the singular part N of the graphexon measure.

In these GXMFG equations on the ring the Lebesgue decomposition of the system graphexon measure $\{G(\alpha, d\beta), \alpha, \beta \in [0, 2\pi)\}$ consists of its absolutely continuous component $\{g(\alpha, \beta), \alpha, \beta \in [0, 2\pi)\}$, which is set to zero, and its singular component $\{N(\alpha, d\beta) = \delta_\alpha, \alpha, \beta \in [0, 2\pi)\}$; this yields

$$\begin{aligned} G(\alpha, d\beta) &= g(\alpha,\beta)d\beta \;+\; N(\alpha, d\beta), \\ &= \delta_\alpha \qquad \alpha, \beta \in [0, 2\pi). \end{aligned} \tag{4}$$

To motivate the second order interaction model in the continuum modeling formulation, consider the finite ring topology and three vertices at

$$(k-1)\Delta\alpha,\; k\Delta\alpha,\; (k+1)\Delta\alpha.$$

Let $m_{k\Delta\alpha}$ denote the state average of the large number of agents residing at node $k\Delta\alpha$. Define

$$\Xi = \frac{1}{(\Delta\alpha)^2}[m_{(k-1)\Delta\alpha}(t) + m_{(k+1)\Delta\alpha}(t) - 2m_{k\Delta\alpha}(t)], \tag{5}$$

where $(\Delta\alpha)^2$ is selected as a scaling parameter so that in the limit Ξ has a well defined value. For subpopulations distributed on a finite ring network, a representative agent at vertex $l = k\Delta\alpha$ has the SDE

$$dx_l(t) = f(l, X_l(t), u_l(t), \mu_l(t))dt + D\Xi dt + \sigma dW_l(t), \tag{6}$$

where Ξ indicates the influence of the neighboring subpopulations.

If $m_\alpha(t)$ has sufficient smoothness with respect to α, we have

$$\Xi \overset{\Delta\alpha \to 0}{\longrightarrow} \partial_\alpha^2 m_\alpha(t)$$

and say that the model has second order spatial differential interaction dynamics. The interested reader is referred to [11] for similar ideas of deriving second order diffusion dynamics based on immediate neighbor interactions in a line network. We consider the case with D being a scalar and moreover $D > 0$. If the immediate neighbour average (in one component of m_α) is higher than at vertex $k\Delta\alpha$, then the state at $k\Delta\alpha$ receives an upward lift along that direction. Such an interaction pattern implies a smoothing effect across the sub-populations in that a subpopulation has a tendency to conform with the neighboring sub-populations.

Employing the model of local influences in the sparse case outlined above and including them in the general GXMFG equations (3) we arrive at the following system representation.

At each vertex $\alpha \in [0, 2\pi)$, a local subpopulation of a large number of agents is situated. We now introduce the following simple nonlinear model for a generic agent at vertex α:

$$dx_\alpha(t) = f(\alpha, x_\alpha(t), u_\alpha(t), \mu_\alpha(t))dt + D\partial_\alpha^2 m_\alpha(t)dt + \sigma dw_\alpha(t), \tag{7}$$

where we have $\alpha \in [0, 2\pi)$, $x_\alpha(t) \in \mathbb{R}^n$, $u_\alpha(t) \in \mathbb{R}^r$, the Brownian motion $w_\alpha(t) \in \mathbb{R}^{r_1}$, and

$$m_\alpha(t) = \int_{\mathbb{R}^n} x\mu_\alpha(t, dx).$$

The function f is allowed to depend on α itself. In the above, $\mu_\alpha(t)$ is the mean field generated by the local subpopulation at vertex α, of which the constituent agents have i.i.d. initial states and independent driving Brownian motions. The second order derivative term $\partial_\alpha^2 m_\alpha(t)$, computed componentwise in $\mathbb{R}^n$, is used to model the impact of the immediate neighbors. We call $\{\mu_\alpha(t)|\alpha \in [0, 2\pi)\}$ a mean field ensemble, as a collection of local mean fields.

Now we further introduce the cost of a generic agent

$$J_\alpha = \mathbb{E}\int_0^T L(x_\alpha(t), u_\alpha(t), \mu_\alpha(t))dt, \tag{8}$$

where for simplicity the terminal cost is taken as zero.

4 The graphexon mean field game equations on a ring

For notational simplicity, we present the graph limit model with scalar individual states and controls, i.e., $n = r = 1$. The Brownian motion is also a scalar. Its extension to the vector case is evident.

We have the HJB equation

$$\begin{aligned}
&[\text{HJB}](\alpha)\\
&-\partial_t V_\alpha(t, x) = \inf_u \Big\{\partial_x V_\alpha(t, x)[f(\alpha, x, u, \mu_\alpha(t)) + D\partial_\alpha^2 m_\alpha(t)]\\
&\qquad\qquad + L(x, u, \mu_\alpha(t))\Big\} + \frac{\sigma^2}{2}\partial_x^2 V_\alpha(t, x),\\
&V_\alpha(T, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad \alpha \in [0, 2\pi).
\end{aligned} \tag{9}$$

The closed-loop state process satisfies the FPK equation

$$\begin{aligned}
&[\text{FPK}](\alpha)\\
&\partial_t p_\alpha(t, x) = -\partial_x\{[f(\alpha, x, u^0, \mu_\alpha(t)) + D\partial_\alpha^2 m_\alpha(t)]p_\alpha(t, x)\} + \frac{\sigma^2}{2}\partial_x^2 p_\alpha(t, x),\\
&[\text{BR}](\alpha) \quad u^0(t, x_\alpha|\mu_G) =: \varphi(t, x_\alpha|\mu_G).
\end{aligned} \tag{10}$$

We use $p_\alpha(t, \cdot)$ to denote the probability density function of the distribution $\mu_\alpha(t)$.

As in standard mean field games, the HJB equation above is derived by solving an optimal control problem of the α agent regarding $\mu_\alpha(\cdot)$ and $m_\alpha(\cdot)$ as fixed. The solution of the HJB-FPK equation system consists of the functions (V_α, p_α) defined on $[0, T] \times \mathbb{R}$, which are indexed by $\alpha \in \mathbb{T}$. The term $m_\alpha(t)$ is determined using $p_\alpha(t, x)$.

5 The Linear Quadratic (LQ) model

On the ring we take $\mathbb{T} = [0, 2\pi)$ as the set of nodes, and the states x_α to lie in $\mathbb{R}^n$. Suppose all initial states have the mean $m_\alpha(0)$ and finite second moment at $t = 0$. We take

$$f(x_\alpha, u_\alpha, \mu_\alpha) = Ax_\alpha + Bu_\alpha + D_0 m_\alpha,$$

where $u_\alpha \in \mathbb{R}^{n_1}$, $m_\alpha = \int_{\mathbb{R}^n} y\mu_\alpha(dy)$ is the mean of the mean field distribution μ_α at node α, and we take

$$L(x_\alpha, u_\alpha, \mu_\alpha) = ||x_\alpha - \Gamma_1 m_\alpha - \Gamma_2\eta||_Q^2 + u_\alpha^T R u_\alpha,$$

where $R > 0$ and $Q \geq 0$, $\|z\|_Q^2 := z^T Q z$, and η is a constant vector. By the above running cost, each agent tries to track a position based on the local population average and a reference position $\eta \in \mathbb{R}^n$.

Now for the best response control problem of the α-agent, we consider the dynamics

$$dx_\alpha(t) =(Ax_\alpha(t) + Bu_\alpha(t) + D_0 m_\alpha(t) + D\partial_\alpha^2 m_\alpha(t))dt + \sigma dw_\alpha(t). \tag{11}$$

The cost of the representative agent is given by

$$\begin{aligned} J_\alpha =&\mathbb{E}\int_0^T (||x_\alpha - \Gamma_1 m_\alpha - \Gamma_2\eta||_Q^2 + u_\alpha^T R u_\alpha)dt \\ &+ \mathbb{E}\|x_\alpha(T) - \Gamma_{1f} m_\alpha(T) - \Gamma_{2f}\eta\|_{Q_f}^2. \end{aligned} \tag{12}$$

The relevant Riccati equation for this MFG is given by

$$0 = \dot{P} + PA + A^T P - PBR^{-1}B^T P + Q, \tag{13}$$

where the terminal condition is $P(T) = Q_f$, and which has a unique solution P on $[0, T]$. We further introduce the linear ODE:

$$\begin{aligned} 0 =&\partial_t S_\alpha(t) + (A^T - PBR^{-1}B^T)S_\alpha(t) \\ &+ P[D_0 m_\alpha(t) + D\partial_\alpha^2 m_\alpha(t)] - Q[\Gamma_1 m_\alpha(t) + \Gamma_2\eta], \end{aligned} \tag{14}$$

where $S_\alpha(T) = -Q_f(\Gamma_{1f} m_\alpha(T) + \Gamma_{2f}\eta)$. Here $S_\alpha(\cdot)$ is viewed as a function of t for the given α.

Lemma 5.1. Suppose $m_\alpha(\cdot)$ is a given function on $[0, T]$ in the optimal control problem (11)–(12). Then the optimal control law is

$$\hat{u}_\alpha(t) = -R^{-1}B^T(P(t)x_\alpha(t) + S_\alpha(t)), \quad 0 \leq t \leq T.$$

5.1 The mean field game equation system

Under the best response control law $\hat{u}_\alpha$, the closed-loop state equation of the α-agent is

$$\begin{aligned} dx_\alpha(t) =&[(A - BR^{-1}BP)x_\alpha(t) - BR^{-1}B^T S_\alpha(t) \\ &+ D_0 m_\alpha(t) + D\partial_\alpha^2 m_\alpha(t)]dt \\ &+ \sigma dw_\alpha(t). \end{aligned} \tag{15}$$

Now we average the states of a large number of agents at vertex α. By the independence of the state processes such an averaging regenerates the same quantity $m_\alpha(t)$. So taking expectations on both sides of (15) yields the equation of evolution of the state mean:

$$\partial_t m_\alpha(t) =[A - BR^{-1}B^T P(t) + D_0]m_\alpha(t) + D\partial_\alpha^2 m_\alpha(t) - BR^{-1}B^T S_\alpha(t),$$

where the initial condition is $m_\alpha(0)$. To determine the solution in the LQ case, it is sufficient to use the equation of m_α instead of the FPK equation.

We summarize the mean field game equations on $\mathbb{T} = [0, 2\pi)$ as follows

$$\begin{aligned} \partial_t S_\alpha(t) =& -(A^T - PBR^{-1}B^T)S_\alpha(t) \\ &+ (Q\Gamma_1 - PD_0)m_\alpha(t) - PD\partial_\alpha^2 m_\alpha(t) + Q\Gamma_2\eta, \end{aligned} \tag{16}$$

$$\partial_t m_\alpha(t) =(A - BR^{-1}B^T P + D_0)m_\alpha(t) + D\partial_\alpha^2 m_\alpha(t) \\ - BR^{-1}B^T S_\alpha(t), \tag{17}$$

where $m_\alpha(0)$ is given and $S_\alpha(T) = -Q_f(\Gamma_{1f} m_\alpha(T) + \Gamma_{2f}\eta)$. The equations above constitute a PDE system with an initial condition and a terminal condition, as a generalization of the standard forward-backward ODE system in mean field games with symmetric players; see [14]. For convenience of further analysis, we may use the alternative notation $S(t, \alpha)$, $m(t, \alpha)$.

The solution of the mean field game in the current case reduces to finding two $\mathbb{R}^n$-valued functions S, m defined on $\mathbf{Q} = [0, T] \times \mathbb{T}$ satisfying (16) and (17).

6 The Linear Quadratic (LQ) model: Infinite rectangular network case

Recall from Section II that the vertexon limit of a uniformly distributed rectangular (i.e. square) grid in $[0, 1]^2$ is a uniform unit density on $[0, 1]^2$, which is evidently not singular, while the corresponding graphexon is given by the sum of two singular measures supported on two foliations of $[0, 1]^4$ namely

$$\Big\{\frac{1}{2}\delta(p - x)\delta(q - y) + \frac{1}{2}\delta(p - y)\delta(q - x);\ 0 \le x, y, p, q \le 1\Big\}.$$

Since no differentiation can occur except in the two directions parallel to the edges of the unit square $[0, 1]^2$, the limit of second order interactions in this case is given by the Laplacian operator

$$\Delta m_{\alpha,\beta}(t) \ = \ [\partial_\alpha^2 \ + \ \partial_\beta^2] m_{\alpha,\beta}(t).$$

And the resulting model for the dynamics of a generic agent at vertex α, β is

$$dx_{\alpha,\beta}(t) =f_0(\alpha, \beta, x_{\alpha,\beta}(t), u_{\alpha,\beta}(t), \mu_{\alpha,\beta}(t))dt \\ + D[\partial_\alpha^2 \ + \ \partial_\beta^2] m_{\alpha,\beta}(t))dt + \sigma dw_{\alpha,\beta}(t), \tag{18}$$

Since the Laplacian here has a positive definite symbol, the existence and uniqueness analysis in the subsequent section for the single network variable case applies to this more complex network case when the following conditions are imposed: (i) the dynamical state dimension is kept to one, and (ii) the dynamical periodic boundary conditions used for the ring topology are employed in the infinite lattice case by making it into a two dimensional torus by identifying opposite edges in the standard way.

Furthermore, it can be seen that this formulation generalizes yet further to infinite rectangular networks of higher dimension.

7 Existence and uniqueness analysis

To analyze the PDE system (16)–(17). We consider the scalar case, i.e. $n = 1$. Accordingly, the state process in (11) is a scalar. We make the following assumption.
Assumption 1. For the model (11), we have $D > 0$.

The condition $D > 0$ is important in existence analysis of the mean field game since (17) becomes a parabolic equation (with spatial variable α) subject to an initial condition.

For a suitably general analysis yielding the existence of classical solutions we introduce certain Hölder spaces. We fix $\delta \in (0, 1)$ to be used in the Hölder norm. For two points x and y on the ring $\mathbb{T}$ parameterized by $[0, 2\pi)$, we define the distance

$$d_r(x, y) = \min\{2\pi - |x - y|, |x - y|\}. \tag{19}$$

Next, on the set $\mathbf{Q} = [0,T]\times\mathbb{T}$, we define the parabolic distance between two points (t,x) and (s,y) as

$$d((t,x),(s,y)) = d_r(x,y) + |t-s|^{1/2}.$$

Based on the distance d we define the Hölder semi-norm for a function $v(t,\alpha)$ defined on $\mathbf{Q}$:

$$[v]_{\delta/2,\delta;\mathbf{Q}} = \sup_{z_i\in\mathbf{Q}, z_1\neq z_2} \frac{|v(z_1)-v(z_2)|}{d^\delta(z_1,z_2)}.$$

Denote $|v|_{0;\mathbf{Q}} = \sup_{z\in\mathbf{Q}}|v(z)|$. We further define the Hölder norms

$$\begin{aligned}|v|_{\delta/2,\delta;\mathbf{Q}} =& |v|_{0;\mathbf{Q}} + [v]_{\delta/2,\delta;\mathbf{Q}},\\ |v|_{1+\delta/2,2+\delta;\mathbf{Q}} =& |v|_{0;\mathbf{Q}} + |\partial_t v|_{0;\mathbf{Q}} + |\partial_\alpha v|_{0;\mathbf{Q}}\\ &+ |\partial_\alpha^2 v|_{0;\mathbf{Q}} + [v]_{1+\delta/2,2+\delta;\mathbf{Q}},\end{aligned}$$

where $[v]_{1+\delta/2,2+\delta;\mathbf{Q}} = [\partial_t v]_{\delta/2,\delta;\mathbf{Q}} + [\partial_\alpha^2 v]_{\delta/2,\delta;\mathbf{Q}}$. We use $C^{\delta/2,\delta}(\mathbf{Q})$ (resp., $C^{1+\delta/2,2+\delta}(\mathbf{Q})$) to denote the space consisting of functions with $|v|_{\delta/2,\delta;\mathbf{Q}} < \infty$ (resp., $|v|_{1+\delta/2,2+\delta;\mathbf{Q}} < \infty$). For a function defined on the ring $\mathbb{T} = [0,2\pi)$, we similarly define the Hölder norm $|v|_{2+\delta,\mathbb{T}}$ using the distance d_r; we may visualize v as a periodic function defined on $\mathbb{R}$. In our further analysis, we will drop $\mathbf{Q}$ from the subscript in the norm without causing confusion.

7.1 The fixed point method

We give some analytical preparation. Consider the following equation [21, Theorem 5.1] [22, Theorem 5.14], [5]

$$\partial_t q(t,\alpha) = D\partial_\alpha^2 q(t,\alpha) + F(t,\alpha)q(t,\alpha) + g(t,\alpha), \tag{20}$$

where $D>0$, $q(0,\alpha)=\varphi(\alpha)$, $\varphi\in C^{2+\delta}(\mathbb{T})$, $g\in C^{\delta/2,\delta}(\mathbf{Q})$, and

$$|F|_{\delta/2,\delta} \le C_1.$$

Remark 7.1. We regard φ as a periodic function with one period $[0,2\pi]$, twice differentiable with a Hölder continuous second derivative.

The following lemma is a corollary to Theorem 5.1 in [21].

Lemma 7.2. Suppose $\mathbf{Q} = [0,T]\times\mathbb{T}$. Then there exists a unique solution $q\in C^{1,2}(\mathbf{Q})$ to (20) and

$$|q|_{1+\delta/2,2+\delta} \le C_0(|g|_{\delta/2,\delta} + |\varphi|_{2+\delta}), \tag{21}$$

where the constant C_0 only depends on (D, C_1, T, δ). □

We proceed to the existence analysis of (16)–(17) by formulating the following fixed point problem. Given a general function $\hat{S}\in C^{\delta/2,\delta}(\mathbf{Q})$ in place of S in (17), by Lemma 7.2, we obtain a unique solution $\hat{m}$ and define the following mapping

$$\hat{m} = \Lambda_1(\hat{S}),$$

which is well defined from $C^{\delta/2,\delta}(\mathbf{Q})$ to $C^{1+\delta/2,2+\delta}(\mathbf{Q})$.

Next with any $\check{m}\in C^{1+\delta/2,2+\delta}(\mathbf{Q})$ in place of m in (16), we determine a unique solution $\check{S}$ and define the mapping

$$\check{S} = \Lambda_2(\check{m}),$$

which will subsequently be shown to be from $C^{1+\delta/2,2+\delta}(\mathbf{Q})$ to $C^{\delta/2,\delta}(\mathbf{Q})$.

Thus, we have the following fixed point equation

$$S = \Lambda_2\Lambda_1(S), \tag{22}$$

which determines a solution of the PDE system (16)–(17).

Consider $m = \Lambda_1(S)$ and $m' = \Lambda_1(S')$, with identical initial condition data $m_\alpha(0) = m'_\alpha(0)$, where S and S' are any functions from $C^{\delta/2,\delta}(\mathbf{Q})$. By Lemma 7.2, we have

$$|m - m'|_{1+\delta/2,2+\delta} \le (C_0 B^2/R)|S - S'|_{\delta/2,\delta}. \tag{23}$$

Below we show that Λ_2 has its image in $C^{\delta/2,\delta}(\mathbf{Q})$. Let $\Phi(t,\tau)$ be the fundamental solution of the ordinary differential equation $\dot z = -(A^T - PBR^{-1}B^T)z$. We take an arbitrary function $m \in C^{1+\delta/2,2+\delta}(\mathbf{Q})$ in (16) and have

$$\begin{aligned} S_\alpha(t) =& \Phi(t,0)S_\alpha(0) + \int_0^t \Phi(t,\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) \\ &- PD\partial_\alpha^2 m_\alpha(\tau) + Q\Gamma_2\eta]d\tau, \end{aligned} \tag{24}$$

where $S_\alpha(0)$ is to be determined. By the terminal condition in (16), we determine

$$\begin{aligned} S_\alpha(0) =& -\int_0^T \Phi(0,\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau) + Q\Gamma_2\eta]d\tau \\ &- \Phi(0,T)Q_f[\Gamma_{1f}m_\alpha(T) + \Gamma_{2f}\eta]. \end{aligned}$$

Substituting the above $S_\alpha(0)$ into (24), we have

$$\begin{aligned} S_\alpha(t) =& \Lambda_2(m)(t) \\ =& -\int_t^T \Phi(t,\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau) \\ &+ Q\Gamma_2\eta]d\tau - \Phi(t,T)Q_f[\Gamma_{1f}m_\alpha(T) + \Gamma_{2f}\eta]. \end{aligned}$$

Now we calculate the Hölder norm of S. We have

$$\begin{aligned} &|S_\alpha(t) - S_{\alpha'}(t)| \\ &\le \int_t^T |\Phi(t,\tau)| \cdot |Q\Gamma_1 - PD_0| \cdot |m_\alpha(\tau) - m_{\alpha'}(\tau)|d\tau \\ &\quad + \int_t^T |\Phi(t,\tau)| \cdot |PD| \cdot |\partial_\alpha^2 m_\alpha(\tau) - \partial_\alpha^2 m_{\alpha'}(\tau)|d\tau \\ &\quad + |\Phi(t,T)Q_f\Gamma_{1f}| \cdot |m_\alpha(T) - m_{\alpha'}(T)|. \end{aligned} \tag{25}$$

Denote $|h|_0 = |h|_{0;[0,T]} = \sup_{t\in[0,T]} |h(t)|$. Hence by (25), we have

$$\begin{aligned} &\sup_{\alpha\ne\alpha', t\in[0,T]} d_r^{-\delta}(\alpha,\alpha')|S_\alpha(t) - S_{\alpha'}(t)| \\ &\le T \sup_{t,\tau\in[0,T]} |\Phi(t,\tau)| \cdot (|Q\Gamma_1 - PD_0|_0 + |PD|_0) \\ &\quad \times (\sup_{\tau,\alpha,\alpha'} d_r^{-\delta}(\alpha,\alpha')|m_\alpha(\tau) - m_{\alpha'}(\tau)| + [m]_{1+\delta/2,2+\delta}) \\ &\quad + \sup_t |\Phi(t,T)Q_f\Gamma_{1f}| \cdot \sup_{\alpha,\alpha'} d_r^{-\delta}(\alpha,\alpha')|m_\alpha(T) - m_{\alpha'}(T)| \\ &\le \{T \sup_{t,\tau} |\Phi(t,\tau)| \cdot (|Q\Gamma_1 - PD_0|_0 + |PD|_0) \\ &\quad + \sup_t |\Phi(t,T)Q_f\Gamma_{1f}|\} \\ &\quad \times (\pi^{1-\delta}|\partial_\alpha m|_{0;\mathbf{Q}} + [m]_{1+\delta/2,2+\delta}). \end{aligned} \tag{26}$$

Next for $0 \le t < t' \le T$, we have

$$\begin{aligned}
&S_\alpha(t) - S_\alpha(t') \\
&= \int_t^T \Phi(t,\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau)]d\tau \\
&\quad - \int_{t'}^T \Phi(t',\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau)]d\tau \\
&\quad + [\Phi(t',T) - \Phi(t,T)]Q_f[\Gamma_{1f} m_\alpha(T) + \Gamma_{2f}\eta] \\
&= \int_t^{t'} \Phi(t,\tau)[(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau)]d\tau \\
&\quad + \int_{t'}^T [\Phi(t,\tau) - \Phi(t',\tau)][(Q\Gamma_1 - PD_0)m_\alpha(\tau) - PD\partial_\alpha^2 m_\alpha(\tau)]d\tau \\
&\quad + [\Phi(t',T) - \Phi(t,T)]Q_f[\Gamma_{1f} m_\alpha(T) + \Gamma_{2f}\eta].
\end{aligned}$$

We use the differentiability of Φ to estimate

$$\begin{aligned}
&|t-t'|^{-\delta/2} \cdot |S_\alpha(t) - S_\alpha(t')| \\
&\le [T^{1-\delta/2} \sup_{t,\tau} |\Phi(t,\tau)| + T^{2-\delta/2} \sup_{t,\tau} |\partial_t \Phi(t,\tau)|] \\
&\quad \times (|Q\Gamma_1 - PD_0|_0 + |PD|_0)(|m|_{0;\mathbf{Q}} + |\partial_\alpha^2 m|_{0;\mathbf{Q}}) \\
&\quad + T^{1-\delta/2} \sup_t |\partial_t \Phi(t,T)| \cdot Q_f \cdot (|\Gamma_{1f}| \cdot |m|_{0;\mathbf{Q}} + |\Gamma_{2f}\eta|).
\end{aligned} \tag{27}$$

Now we conclude $S = \Lambda_2(m) \in C^{\delta/2,\delta}(\mathbf{Q})$.

Denote the constant

$$\begin{aligned}
K_0 =& \pi^{1-\delta}(T + T^{1-\delta/2} + T^{2-\delta/2}) \sup_{t,\tau\le T} [|\Phi(t,\tau)| + |\partial_t \Phi(t,\tau)|] \\
&\times (|Q\Gamma_1 - PD_0|_0 + |PD|_0 + Q_f|\Gamma_{1f}|) \\
&+ \pi^{1-\delta} \sup_{t,\tau\le T} |\Phi(t,\tau)| \cdot |Q_f\Gamma_{1f}|.
\end{aligned} \tag{28}$$

Now consider $z_1 = (t_1, \alpha_1)$ and $z_2 = (t_2, \alpha_2)$. We have

$$\begin{aligned}
\frac{|S(z_1) - S(z_2)|}{d^\delta(z_1,z_2)} \le& \frac{|S(z_1) - S(t_1,\alpha_2) + S(t_1,\alpha_2) - S(z_2)|}{d^\delta(z_1,z_2)} \\
\le& d_r^{-\delta}(\alpha_1,\alpha_2)|S(z_1) - S(t_1,\alpha_2)| \\
&+ |t_1 - t_2|^{-\delta/2}|S(t_1,\alpha_2) - S(z_2)|,
\end{aligned}$$

which combined with estimates (26) and (27) along the direction of t and α, respectively, implies that

$$\begin{aligned}
|S|_{\delta/2,\delta} \le& K_0|m|_{1+\delta/2,2+\delta} + T^{1-\delta/2} \sup_t |\partial_t \Phi(t,T)| \cdot Q_f \cdot |\Gamma_{2f}\eta| \\
&+ (T|Q\Gamma_2\eta| + |Q_f\Gamma_{2f}\eta|) \sup_{t,\tau} |\Phi(t,\tau)|
\end{aligned} \tag{29}$$

Theorem 7.3. The PDE system (16)–(17) for (S,m) has a unique solution if $C_0K_0B^2/R < 1$, where C_0 is given in Lemma 7.2.

Proof. For two arbitrary functions m, m' in $C^{1+\delta/2,2+\delta}(\mathbf{Q})$, define $S = \Lambda_2(m)$, $S' = \Lambda_2(m')$. Then we follow the method used in establishing (29) to similarly obtain

$$\begin{aligned}
|S - S'|_{\delta/2,\delta} &= |\Lambda_2(m) - \Lambda_2(m')|_{\delta/2,\delta} \\
&\le K_0|m - m'|_{1+\delta/2,2+\delta}.
\end{aligned} \tag{30}$$

So by (30), we have

$$
\begin{aligned}
|\Lambda_2\Lambda_1(S) - \Lambda_2\Lambda(S')|_{\delta/2,\delta} &\le K_0|\Lambda_1(S) - \Lambda_1(S')|_{1+\delta/2,2+\delta} \\
&\le (C_0K_0B^2/R)|S-S'|_{\delta/2,\delta},
\end{aligned}
$$

where the last line follows from (23). The proof follows from a fixed point theorem on the Banach space $C^{\delta/2,\delta}(\mathbf{Q})$. □

Remark 7.4. The contraction condition is satisfied when T is sufficiently small and the system has weak coupling, i.e., $|\Gamma_1| + |\Gamma_{1f}| + |D_0| + |D|$ is sufficiently small.

7.2 Numerical example

Example 7.5. In the LQ model we take $A = 0.5$, $B = 1$, $D_0 = 0.2$, $D = 0.5$, $Q = 2$, $R = 1$, $\Gamma_1 = 0.8$, $\Gamma_2 = \eta = 0$, $Q_f = \Gamma_{1f} = \Gamma_{2f} = 0$, $T = 2$, and $m_\alpha(0) = \alpha(1-\alpha)$, $0 \le \alpha \le 1$.

The system (16)–(17) is solved by a difference scheme, with stepsize $\Delta t = 0.0025$ and $\Delta\alpha = 0.05$. For convenience of numerical computation, the parameterization of the ring is normalized and denoted as $[0,1)$ (instead of $[0,2\pi)$). The numerical solution of S, m is displayed in Fig. 1.

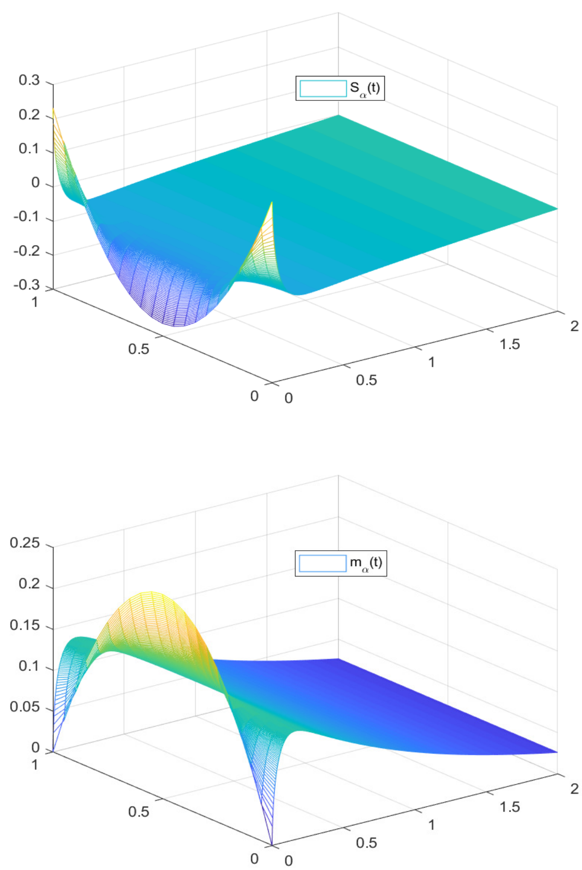

Figure 1: $S_\alpha(t)$ and $m_\alpha(t)$ under the second order diffusive interactions; computed by iterations with $t \in [0,2]$ and $\alpha \in [0,1)$.

8 Conclusion

Future work will address general existence and uniqueness analyses for nonlinear GXMFG systems on a variety of graphexon limits together with their corresponding ϵ-Nash properties.

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