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The routing-and-driving problem for plug-in hybrid electric vehicles

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Abstract : The routing-and-driving problem for plug-in hybrid electric vehicles (PHEVs) is an extension of the vehicle routing problem with time windows, where routing involves determining optimal routes and recharging decisions for a fleet of PHEVs, while driving involves choosing the speed and operating mode on each road segment traveled by a vehicle. Specifically, four driving modes are considered: fuel-only, electricity-only, a combination of fuel and electricity, and energy recuperation, which returns energy to the battery. We consider two variants of the problem where the speed variables are either continuous, which results in a non-linear model, or discrete, which represents the case when speeds are chosen from a predetermined set. To solve these two models, we propose a set of valid inequalities to strengthen the continuous linear programming relaxation and we use a tailored branch-and-cut algorithm. Extensive computational experiments demonstrate the efficiency of the proposed solution methods, which can optimally solve instances with a realistic number of customers and recharging stations. In addition, we show that incorporating speed optimization can significantly reduce the energy consumption costs of a PHEV fleet compared to using fixed speeds.

Keywords : Plug-in hybrid electric vehicle, vehicle routing problem with time windows, speed optimization, branch-and-cut

1 Introduction

Plug-in hybrid electric vehicles (PHEVs) are a promising option for vehicle electrification, as they offer a longer driving range compared to battery electric vehicles (BEVs) and have lower emissions than conventional internal combustion engine (ICE) vehicles. PHEVs operate using two power sources: an ICE running on fuel and an electric machine (EM) powered by electricity. This dual power system allows PHEVs to drive in fuel-only, electricity-only, or combined fuel-and-electric modes. Additionally, when equipped with an energy recuperation system, PHEVs can utilize an energy recuperation mode that returns energy to the battery.

The vehicle routing problem with time windows (VRPTW) is an important generalization of the classical vehicle routing problem, where the objective is to select optimal routes for a fleet of vehicles to serve a set of customers within specified time windows (Cordeau et al. 2007). When the VRPTW involves PHEVs, two additional dimensions must be considered. First, selecting the appropriate driving mode for each road segment. This is crucial because the cost of operating a PHEV depends on the chosen mode (Nejad et al. 2017, Sun and Zhou 2016). Second, optimizing the driving speed on each road segment. Extensive studies (e.g., Bektaş and Laporte 2011, Demir et al. 2014, Fukasawa et al. 2018) have shown that incorporating speed optimization into the routing process has the potential to enhance the accuracy of energy consumption estimation and result in substantial energy savings. Since both the operating mode and speed are driving decisions, we refer to the VRPTW with driving mode choice and speed optimization for PHEVs as the PHEV Routing-and-Driving Problem, which is denoted by PHEVRDP in this paper.

The literature on the PHEV routing problem (PHEVRP) is relatively scarce. Mancini (2017) proposed a PHEVRP where each route can be served either exclusively by an EM or partially by utilizing traditional fuel propulsion. In their setting, the cost of using electricity is considerably lower than that of using fuel. To address this problem, they introduced a matheuristic based on a large neighborhood search strategy. Vincent et al. (2017) formulated a PHEVRP that allows the vehicle to choose between fuel-only or electricity-only mode on each road segment. They solved the problem using a simulated annealing heuristic. Hiermann et al. (2019) presented a VRPTW that encompasses various vehicle types, including PHEVs, BEVs, and ICE vehicles (ICEVs). To solve this problem, they developed an algorithm that combines a genetic algorithm with local and large neighborhood search techniques. Bahrami et al. (2020) integrated a power management model with the PHEVRP. They first optimized the power distribution between electricity and fuel for each arc and subsequently solved the PHEVRP. To address this problem, they proposed a branch-and-price algorithm along with a heuristic approach. Zhen et al. (2020) assumed that a PHEV can operate in four modes, including battery-based, gasoline-based, balanced (battery and gasoline), and gasoline-only modes. They proposed a PHEVRP that incorporates mode selection on each road segment. Masmoudi et al. (2022) conducted a study on a waste collection problem involving a fleet of PHEVs powered by electricity and compressed natural gas (CNG). In this problem, the vehicles have the flexibility to select either electricity or CNG as their power source on each arc. To solve this problem, they devised a hybrid threshold acceptance metaheuristic. To address large-scale instances of the problem, they developed a heuristic called improved particle swarm optimization.

Another relevant area of literature focuses on hybrid electric vehicles (HEVs), which differ from PHEVs in that they cannot recharge their battery from an external electrical outlet. However, they possess the ability to recharge their battery through the use of fuel. In the early work by Doppstadt et al. (2016), they considered four operating modes for HEVs: pure combustion, pure electric, charging the battery while driving in combustion mode, and a boost mode combining the EM and ICE. They integrated mode selection with the travelling salesman problem (TSP) and proposed a tabu search heuristic to solve the problem. This problem was later extended to include customer time windows (Doppstadt et al. 2020, Rocha and Subramanian 2023). In addition to the TSP, the shortest path problem has also been explored for HEVs (e.g., Liu et al. 2019, De Nunzio et al. 2021).

In addition to PHEVs and HEVs, BEVs are another prominent direction in vehicle electrification, and the associated routing problems have received significant attention in the existing literature (e.g., Desaulniers et al. 2016, Florio et al. 2021, Montoya et al. 2017, Pelletier et al. 2019, Schneider et al. 2014). For a comprehensive overview of the BEV routing literature, we refer readers to the work of Pelletier et al. (2016). Compared to PHEVs and HEVs, BEVs often have a shorter driving range due to their limited battery capacity. Consequently, the BEV routing problem requires accurate monitoring of energy consumption to prevent running out of battery power along the route. To calculate the energy consumption of a BEV, previous studies such as Felipe et al. (2014), Hiermann et al. (2016), and Schneider et al. (2014) assume a linear relationship between energy consumption and travel distance. However, it should be noted that the energy consumption of a BEV is influenced not only by the driving distance but also by factors such as driving speed, road gradient, and cargo load. Some studies, such as Goeke and Schneider (2015), Lebeau et al. (2015), and Pelletier et al. (2019), assume a constant driving speed on each arc and calculate the energy consumption accordingly.

Table 1 presents a summary of the existing literature. We can see that the majority of previous studies in this field assumed a predetermined driving speed for each arc, overlooking the potential benefits of adjusting driving speed to reduce energy consumption. To the best of our knowledge, Wu et al. (2024) were the first to integrate speed optimization into a PHEV TSP without time windows, and with charging stations located at customer locations. However, the integration of speed optimization and VRPTW for PHEVs remains an unexplored area. The present research addresses this gap by introducing the PHEVRDP, which not only combines the VRPTW with speed optimization and operating mode decisions, but also considers charging stations located anywhere. A straightforward approach to formulating the PHEVRDP is by extending the TSP model proposed in Wu et al. (2024) to a VRPTW with a vehicle index. As will be shown in the computational experiments, this approach proves inefficient. Therefore, the primary contribution of this research is the development of models for the PHEVRDP that omit the vehicle index. While this introduces additional challenges, such as the inability to track the status of each PHEV, including state of charge and travel time, it significantly improves computational efficiency compared to the vehicle-indexed formulation.

Table 1: Comparison of the problem characteristics studied in this paper and in related TSP and VRP papers

Papers	Vehicles	Problem	Time windows	Charging stations	Speed optimization	Solution method
Bahrami et al. (2020)	PHEV	VRP	–	√	–	E & H
Desaulniers et al. (2016)	BEV	VRP	√	√	–	E
Doppstadt et al. (2016)	HEV	TSP	–	–	–	H
Doppstadt et al. (2020)	HEV	TSP	√	–	–	H
Felipe et al. (2014)	BEV	VRP	–	√	–	H
Goeke and Schneider (2015)	BEV	VRP	√	√	–	H
Hiermann et al. (2016)	BEV	VRP	√	√	–	E & H
Hiermann et al. (2019)	PHEV, BEV, ICEV	VRP	√	√	–	H
Lebeau et al. (2015)	BEV	VRP	√	√	–	H
Mancini (2017)	PHEV	VRP	–	√	–	H
Masmoudi et al. (2022)	PHEV	VRP	√	√	–	H
Pelletier et al. (2019)	BEV	VRP	–	–	–	E & H
Rocha and Subramanian (2023)	HEV	TSP	√	–	–	H
Schneider et al. (2014)	BEV	VRP	√	√	–	H
Vincent et al. (2017)	PHEV	VRP	–	√	–	H
Wu et al. (2024)	PHEV	TSP	–	√	√	E
Zhen et al. (2020)	PHEV	VRP	–	√	–	H
This paper	PHEV	VRP	√	√	√	E & H

'–' means that the paper does not consider the feature and '√' means that the paper considers the feature; In column 'Solution method', E and H denote exact and heuristic algorithms, respectively.

Compared to the VRP with speed optimization (VRPS) for traditional fuel vehicles, initially proposed by Bektaş and Laporte (2011), the PHEVRDP without recharging possibilities is more complex because it requires the additional step of selecting operating modes. However, the VRPS is already quite challenging, often solved heuristically in the relevant literature (e.g., Demir et al. 2012, 2014). Fukasawa et al. (2018) proposed a branch-cut-and-price algorithm for solving the VRPS exactly; however, they were unable to solve some instances with 20 customers and they could solve just over half of the instances with 25 customers. To efficiently solve the PHEVRDP, this paper proposes valid inequalities for the models and incorporates them into a branch-and-cut algorithm. Specifically, when recharging is not considered, the proposed method can optimally solve all instances with 50 customers and over 80 percent of the instances with 60 or 70 customers for both the continuous and discrete speed variable models within 2 hours of computing time. However, when recharging possibilities are taken into account, the problem becomes significantly more complex due to the need to select recharging stations for the PHEVs. The proposed method can optimally solve the model with continuous speed variables for nearly all instances with 40 customers and four recharging stations, except for one, and can optimally solve the model with discrete speed variables for all instances with 40 customers and four recharging stations within 2 hours.

In summary, our contributions to the existing literature are fourfold. First, we formally define the PHEVRDP for a fleet of PHEVs, which simultaneously determines optimal routes, recharging strategies, and the speeds and driving modes along these routes. Specifically, we consider two variants of the PHEVRDP: one with continuous speed variables and another one with discrete speed variables based on a predetermined set of speed choices. Second, we develop two models for these variants without a vehicle index: a mixed-integer non-linear programming (MINLP) model for the continuous speed variant and a mixed-integer linear programming (MILP) model for the discrete speed variant. Additionally, we enhance the continuous linear programming relaxation of both models by introducing a set of valid inequalities. Third, to efficiently solve these problems, we develop a tailored branch-and-cut algorithm alongside an iterated optimization heuristic. Fourth, we conduct extensive computational experiments to assess the proposed models and solution methods, including a comparison with a vehicle index formulation extended from a previous work by Wu et al. (2024), demonstrating their effectiveness in solving instances with a realistic number of customers and recharging stations. Finally, to promote reproducibility and facilitate knowledge sharing, we have made our code and instances publicly accessible.

The remainder of the paper is structured as follows. Section 2 provides a formal definition of the PHEVRDP and develops the corresponding mixed-integer programming models. Section 3 introduces a series of valid inequalities designed to strengthen the proposed models. Section 4 describes a branch-and-cut algorithm and an iterated optimization heuristic tailored for solving the proposed models. Section 5 presents the results of numerical experiments conducted to evaluate the performance of the proposed models and solution methods. Finally, Section 6 concludes the paper.

2 Problem definition

In this section, we begin by introducing an energy consumption model that is predicated on vehicle speed. Subsequently, we describe an MINLP model for the PHEVRDP with continuous speed variables. Finally, we describe an MILP model for the PHEVRDP with discrete speed variables.

2.1 Energy consumption model

The total tractive power P_{tra} (watts) usage of a PHEV can be calculated by multiplying the speed by the tractive force, which is composed of inertial force, grade resistance, air drag resistance, and rolling resistance, as follows:

$$P_{tra} = v_t \left(ma_t + mg\sin\theta_t + \frac{1}{2}C_d\rho F v_t^2 + C_r mg\cos\theta_t \right), \tag{1}$$

where v_t is the instantaneous speed (meter/second), m is the curb-weight (kilogram), a_t is the instantaneous acceleration (meter/second^2), g is the gravitational constant (meter/second^2), θ_t is the road gradient (radian), C_d is the coefficient of aerodynamic drag, ρ is the air density (kilogram/meter^3), F is the frontal surface area (meter^2), and C_r is the coefficient of rolling resistance. Tractive power model (1) is consistent with the models used in the existing PHEV studies (e.g., Graver et al. 2011, Wu et al. 2024), BEV studies (e.g., Asamer et al. 2016, Fiori et al. 2016), and diesel vehicles studies (e.g., Barth et al. 2005).

In the joint routing and speed optimization studies, vehicles are usually assumed to be running at a constant speed along each road segment (e.g., Baum et al. 2020, Bektaş and Laporte 2011, Demir et al. 2014, Fukasawa et al. 2018). This assumption is not restrictive, as it permits the inclusion of intermediate nodes to represent varying conditions. Therefore, we adhere to this assumption and proceed to calculate the tractive energy demand over a road segment $[0, S]$ as follows:

$$E_{tra}(v) = P_{tra}\frac{S}{v} = \left(mg\sin\theta + \frac{1}{2}C_d\rho F v^2 + C_r mg\cos\theta\right)S, \tag{2}$$

where v is the average speed (meter/second). Note that $E_{tra}(v)$ can be negative when the vehicle is braking or driving downhill, allowing the vehicle to recharge its battery using a motor generator.

Finally, the energy usage can be calculated as the summation of the energy requirement (positive) and the energy recovery (negative) (Murakami 2017, Yi and Bauer 2018), as shown below:

$$\begin{aligned} E(v) &= \frac{1}{\eta_d}\max\{E_{tra}(v), 0\} + \eta_g \min\{E_{tra}(v), 0\} \\ &= \left(\frac{1}{\eta_d} - \eta_g\right)\max\{E_{tra}(v), 0\} + \eta_g E_{tra}(v), \end{aligned} \tag{3}$$

where η_d denotes the drivetrain efficiency and η_g denotes the regeneration efficiency. Typical values for the parameters in model (3) can be found in Table 2.

Table 2: Description of the parameters in equation (3) and their typical values (Demir et al. 2014)

Notation	Description	Typical value
m	Curb-weight (kilogram)	6350
g	Gravitational constant (meter/second^2)	9.81
C_d	Coefficient of aerodynamic drag	0.7
ρ	Air density (kilogram/meter^3)	1.2041
F	Frontal surface area (meter^2)	3.912
C_r	Coefficient of rolling resistance	0.01

2.2 The PHEV routing-and-driving problem with continuous speed optimization

Let $N = \{1, 2, ..., n\}$ be the set of customers, R be the set of recharging stations, and vertices 0 and $n+1$ represent the starting and ending depots, respectively. In line with Schneider et al. (2014), we assume that each recharging station can be visited multiple times by any vehicle. To account for these visits, we introduce a set of dummy vertices, denoted by R', which represents the visits to recharging stations. Let $V' = N \cup R' \cup \{0, n+1\}$ denote the set of vertices and $A = \{(i,j) | i, j \in V', i \neq j\}$ denote all possible connections between the vertices. This allows us to represent the problem on a complete directed graph $G = (V', A)$. Let $N_0 = \{0\} \cup N$ be the set of the starting depot and customers and let $K = \{1, 2, ..., |K|\}$ be a group of homogeneous PHEVs positioned at the starting depot 0. Additionally, we use $\delta^+(\Omega)$ to denote the set of arcs with head nodes in Ω and tail nodes outside of Ω, $\delta^-(\Omega)$ to denote the set of arcs with tail nodes in Ω and head nodes outside of Ω, and $D(\Omega)$ to denote the set of arcs with both nodes in Ω. For simplicity, we use δ_i^+ to represent the set of arcs with head node i, and δ_i^- to represent the set of arcs with tail node i. Following a common simplification in VRP studies (e.g., Desaulniers et al. 2016, Schneider et al. 2014), we assume that the load of the vehicle is negligible in comparison to the curb weight (e.g., parcel delivery).

Each arc $(i, j) \in A$ is characterized by its distance d_{ij}, road gradient θ_{ij}, lower bound on speed $\underline{v}_{ij}$, and upper bound on speed $\overline{v}_{ij}$. Each customer $i \in N$ is assigned a time window $[\underline{\tau}_i, \overline{\tau}_i]$ during which the service should start. Each vehicle needs to arrive at the ending depot $n+1$ by the specified time $\overline{\tau}_{n+1}$. In addition, each customer has a positive demand q_i. Similar to the settings employed in previous studies such as those of Desaulniers et al. (2016) and Hiermann et al. (2019), we assume a linear relationship between the amount of recharging energy and the charging time. To simplify the model, we introduce the parameter ϵ_i to represent the recharging rate at each node $i \in V'$, where $\epsilon_i \geq 0$ for recharging stations and $\epsilon_i = 0$ for customers and depots. Each PHEV is associated with a maximum capacity Q and an initial battery charge level $\overline{B}$, subject to the constraint that the battery charge level must not fall below the minimum battery charge level $\underline{B}$.

For the decision variables, let x_{ij} be a binary variable taking value 1 if and only if a vehicle travels on arc (i, j), and x^f_{ij} be a binary variable taking value 1 if and only if a vehicle is operating in fuel-only mode on arc (i, j). Similary, x^e_{ij}, x^b_{ij}, and x^r_{ij} take value 1 if and only if a vehicle is operating in electricity-only mode, boost mode (combining fuel and electric), or energy recuperation mode on arc (i, j), respectively. Finally, s_i is the state of charge of the vehicle when it visits node i, t_i is the time at which a vehicle visits node i, and w_i is the recharging time of a vehicle at recharging station i.

Our goal is to minimize the total cost of energy consumption for the entire fleet. Let c_f, c_e, c_b, and $-c_e$ represent the unit costs of energy consumption in fuel-only, electric-only, boost, and energy recuperation modes, respectively, and μ be the coefficient of the electricity energy split in the boost mode. Since electricity is cheaper than fuel, we assume $c_e \leq c_b \leq c_f$. The objective function can then be formulated as follows:

$$\min \sum_{(i,j)\in A} \left(c_f x^f_{ij} + c_e x^e_{ij} + c_b x^b_{ij} + c_e x^r_{ij} \right) E_{ij}, \tag{4}$$

where the energy consumption E_{ij} will be negative in energy recuperation mode.

First, we assume that the vehicle can choose only one driving mode per road segment. This is not restrictive, as intermediate nodes can be added if necessary. This results in the following constraints:

$$x^f_{ij} + x^e_{ij} + x^b_{ij} + x^r_{ij} = x_{ij} \qquad \forall (i,j) \in A \tag{5}$$

$$\begin{aligned} E_{ij} = {} & \eta_g d_{ij} x_{ij} \left(mg \sin\theta_{ij} + \frac{1}{2} C_d \rho F v^2_{ij} + C_r mg \cos\theta_{ij} \right) \\ & + \left(\frac{1}{\eta_d} - \eta_g \right) d_{ij} x_{ij} \max\left\{ mg \sin\theta_{ij} + \frac{1}{2} C_d \rho F v^2_{ij} + C_r mg \cos\theta_{ij}, 0 \right\} \qquad \forall (i,j) \in A \end{aligned} \tag{6}$$

$$(x^f_{ij} + x^e_{ij} + x^b_{ij} - 1) M_{ij} \leq E_{ij} \qquad \forall (i,j) \in A \tag{7}$$

$$(1 - x^r_{ij}) M_{ij} \geq E_{ij} \qquad \forall (i,j) \in A \tag{8}$$

$$x_{ij}, x^f_{ij}, x^e_{ij}, x^b_{ij}, x^r_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A \tag{9}$$

$$\underline{v}_{ij} \leq v_{ij} \leq \overline{v}_{ij} \qquad \forall (i,j) \in A, \tag{10}$$

where $M_{ij}, (i, j) \in A$, are large constants that can be set as the maximum absolute value of E_{ij}. Constraints (5) ensure that each arc traveled by a vehicle has a single driving mode. Constraints (6) calculate the energy consumption of each vehicle on each arc, which will be 0 if arc (i, j) is not in the optimal route. Constraints (7) require that fuel-only, electric-only, and boost modes are not chosen under a negative energy consumption. Constraints (8) enforce that the energy recuperation mode cannot be chosen under a positive energy consumption. Constraints (9) define the binary variables. Constraints (10) limit the speed of each vehicle across the entire network. In this formulation, we assume that $\underline{v} > 0$ to ensure the feasibility of terms $d_{ij}/v_{ij}, (i, j) \in A$, as introduced in constraints (22) later.

Second, since the PHEV has a battery, we must monitor the state of charge for each vehicle to ensure it stays within battery limits. This results in the following constraints:

$$s_i + \epsilon_i w_i - (x^e_{ij} + \mu x^b_{ij} + x^r_{ij})E_{ij} \geq s_j - (1 - x_{ij})S_{ij} \qquad \forall (i,j) \in A \backslash \delta^-_{n+1} \tag{11}$$

$$s_i + \epsilon_i w_i - (x^e_{ij} + \mu x^b_{ij} + x^r_{ij})E_{ij} \leq s_j + (1 - x_{ij})S_{ij} \qquad \forall (i,j) \in A \backslash \delta^-_{n+1} \tag{12}$$

$$s_i + \epsilon_i w_i - (x^e_{ij} + \mu x^b_{ij})E_{ij} \geq 0 \qquad \forall (i,j) \in \delta^-_{n+1} \tag{13}$$

$$s_i + \epsilon_i w_i - x^r_{ij} E_{ij} \leq \overline{B} \qquad \forall (i,j) \in \delta^-_{n+1} \tag{14}$$

$$\underline{B} \leq s_i \leq \overline{B} \qquad \forall i \in V' \tag{15}$$

$$w_i \geq 0 \qquad \forall i \in V', \tag{16}$$

where $S_{ij}, (i,j) \in A$, are large constants that can be set as $\overline{B} + M_{ij}$. Constraints (11)–(12) calculate the state of charge of a vehicle at each node except the ending depot. Constraints (13)–(14) restrict the state of charge of each vehicle at the ending depot to be within the range of 0 to $\overline{B}$. Constraints (15) impose limits on each vehicle's battery charge level, constraining it within the minimum and maximum charge levels. Constraints (16) ensure that the recharging time at each node is non-negative.

Third, we incorporate standard flow constraints, time windows, and subtour elimination constraints, consistent with typical VRPTW formulations:

$$\sum_{(i,j)\in\delta^+_0} x_{ij} \leq |K| \tag{17}$$

$$\sum_{(i,j)\in\delta^+_0} x_{ij} \geq \left\lceil \frac{\sum_{i\in N} q_i}{Q} \right\rceil \tag{18}$$

$$\sum_{(i,j)\in\delta^+_i} x_{ij} = 1 \qquad \forall i \in N \tag{19}$$

$$\sum_{(i,j)\in\delta^+_i} x_{ij} \leq 1 \qquad \forall i \in R' \tag{20}$$

$$\sum_{(i,j)\in\delta^+_i} x_{ij} - \sum_{(j,i)\in\delta^-_i} x_{ji} = 0 \qquad \forall i \in N \cup R'. \tag{21}$$

$$t_i + w_i + \frac{d_{ij}}{v_{ij}} - t_j \leq (1 - x_{ij})T_{ij} \qquad \forall (i,j) \in A \tag{22}$$

$$\underline{\tau}_i \leq t_i \leq \overline{\tau}_i \qquad \forall i \in V' \tag{23}$$

$$\sum_{(i,j)\in\delta^+(\Omega)} x_{ij} \geq \Delta(\Omega) \qquad \forall \Omega \subseteq N \cup R', \Omega \neq \emptyset \tag{24}$$

$$\sum_{(i,j)\in D(\Omega)} x_{ij} \leq |\Omega| - \sum_{(k,j)\in\delta^+_k} x_{kj} \qquad \forall \Omega \subseteq N \cup R', k \in \Omega \cap R', \tag{25}$$

where $T_{ij}, (i,j) \in A$, are large constants that can be set as $\max\left\{\overline{\tau}_i + \frac{\overline{B}-\underline{B}}{\epsilon_i} + \frac{d_{ij}}{\underline{v}_{ij}} - \underline{\tau}_j, 0\right\}$, $\Delta(\Omega)$ is the minimum number of vehicle routes needed to serve Ω and can be calculated as $\Delta(\Omega) = \left\lceil \frac{\sum_{i\in\Omega} q_i}{Q} \right\rceil$. Constraints (17)–(18) define the upper and lower bounds on the number of vehicles departing from the depot, respectively. Constraints (19) ensure that each customer is visited by exactly one vehicle, guaranteeing that all demands are satisfied. Constraints (20) restrict each recharging station's visit to at most once. Constraints (21) define the flow constraints. Constraints (22)–(23) are time windows. Constraints (24) act as capacity constraints and effectively eliminate any subtours between customers. In a similar way to constraints (58) in Adulyasak et al. (2014), we introduce constraints (25) to eliminate undesired subtours between recharging stations and customers.

In the remainder of this paper, we will refer to the above MINLP model, consisting of the objective function (4) and constraints (5)–(25), as the PHEVRDP with continuous speed optimization

(PHEVRDP-C). This formulation, without a vehicle index, shows greater computational efficiency compared to the formulation with a vehicle index presented in Appendix A, as observed in our computational experiments.

2.3 The PHEV routing-and-driving problem with discrete speed optimization

Given that drivers in real-world situations typically adjust vehicle speed in discrete intervals, and that a vehicle's cruise control system is similarly designed to operate with discrete speed levels, a vehicle on the road usually has limited speed choices. We assume here that the possible speed levels along each arc (i, j) are described by a set $\{\nu_{ij1}, \nu_{ij2}, ..., \nu_{ij,|L_{ij}|}\}$. When a vehicle is running at a speed level ν_{ijl} on arc (i, j), the energy consumption E_{ijl} can be calculated as follows:

$$\begin{aligned} E_{ijl} =& \eta_g d_{ij} \left(mg \sin\theta_{ij} + \frac{1}{2} C_d \rho F \nu_{ijl}^2 + C_r mg \cos\theta_{ij} \right) \\ &+ \left(\frac{1}{\eta_d} - \eta_g \right) d_{ij} \max \left\{ mg \sin\theta_{ij} + \frac{1}{2} C_d \rho F \nu_{ijl}^2 + C_r mg \cos\theta_{ij}, 0 \right\}, \end{aligned} \tag{26}$$

and the travel time can be calculated as d_{ij}/ν_{ijl}. Note that in Bektaş and Laporte (2011), this speed set is also used to discretize a continuous speed range $[\epsilon, \overline{v}_{ij}]$, where $\nu_{ij1} = \epsilon$, $\nu_{ij,|L_{ij}|} = \overline{v}_{ij}$, and $L_{ij} = \{1, 2, ..., |L_{ij}|\}$.

For the decision variables, let x^f_{ijl} be a binary variable taking value 1 if and only if a vehicle is running at speed level ν_{ijl} and operating in fuel mode on arc (i, j). Similarly, x^e_{ijl}, x^b_{ijl}, and x^r_{ijl} take value 1 if and only if a vehicle is running at speed level ν_{ijl} and operating in electric, boost, or energy recuperation mode on arc (i, j), respectively. Then the problem can be formulated as the following MILP model:

$$\min \quad \sum_{(i,j)\in A} \sum_{l\in L_{ij}} \left(c_f x^f_{ijl} + c_e x^e_{ijl} + c_b x^b_{ijl} + c_e x^r_{ijl} \right) E_{ijl} \tag{27}$$

$$\text{s.t.} \quad \sum_{l\in L_{ij}} \left(x^f_{ijl} + x^e_{ijl} + x^b_{ijl} + x^r_{ijl} \right) = x_{ij} \qquad \forall (i,j) \in A \tag{28}$$

$$s_i + \epsilon_i w_i - \sum_{l\in L_{ij}} \left(x^e_{ijl} + \mu x^b_{ijl} + x^r_{ijl} \right) E_{ijl} \geq s_j - (1 - x_{ij}) S_{ij} \qquad \forall (i,j) \in A \backslash \delta^-_{n+1} \tag{29}$$

$$s_i + \epsilon_i w_i - \sum_{l\in L_{ij}} \left(x^e_{ijl} + \mu x^b_{ijl} + x^r_{ijl} \right) E_{ijl} \leq s_j + (1 - x_{ij}) S_{ij} \qquad \forall (i,j) \in A \backslash \delta^-_{n+1} \tag{30}$$

$$s_i + \epsilon_i w_i - \sum_{l\in L_{ij}} \left(x^e_{ijl} + \mu x^b_{ijl} \right) E_{ijl} \geq 0 \qquad \forall (i,j) \in \delta^-_{n+1} \tag{31}$$

$$s_i + \epsilon_i w_i - \sum_{l\in L_{ij}} x^r_{ijl} E_{ijl} \leq \overline{B} \qquad \forall (i,j) \in \delta^-_{n+1} \tag{32}$$

$$t_i + w_i + \sum_{l\in L_{ij}} \left(x^f_{ijl} + x^e_{ijl} + x^b_{ijl} + x^r_{ijl} \right) \frac{d_{ij}}{v_{ijl}} - t_j \leq (1 - x_{ij}) T_{ij} \qquad \forall (i,j) \in A \tag{33}$$

$$x^f_{ijl}, x^e_{ijl}, x^b_{ijl}, x^r_{ijl} \in \{0, 1\} \qquad \forall (i,j) \in A, \forall l \in L_{ij} \tag{34}$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A \tag{35}$$

(15)–(21), (23)–(25).

Constraints (28) ensure that each arc traveled by a vehicle has a single driving mode and speed level. Constraints (29)–(32) are counterparts of constraints (11)–(14). Constraints (33) calculate the arrival time of each node. Constraints (34)–(35) define the binary variables. In the following sections, we will refer to the above model as the PHEVRDP with discrete speed optimization (PHEVRDP-D).

3 Valid inequalities

This section introduces several families of valid inequalities to strengthen the models proposed in the previous section. Given the relatively small number of these inequalities, they can be directly incorporated into the models as standard constraints.

3.1 Electricity cover inequalities

Given that PHEVRDP-C does not calculate each vehicle's state of charge at the ending depot, we introduce new variables $s_i^* \geq 0, i \in N \cup R'$, subject to the following constraints:

$$s_i^* + (1 - x_{ij})S_{ij} \geq s_i + \epsilon_i w_i - (x_{ij}^e + \mu x_{ij}^b + x_{ij}^r)E_{ij} \qquad \forall (i,j) \in \delta_{n+1}^-. \tag{36}$$

These variables represent the state of charge at the ending depot in the optimal routes, and they are assigned a value of zero otherwise. We use A^* to denote the optimal routes of PHEVRDP-C, namely $x_{ij} = 1, \forall (i,j) \in A^*$. We can then aggregate constraints (11) over A^*, resulting in the following inequality:

$$\sum_{(i,j)\in A^*} (x_{ij}^e + \mu x_{ij}^b + x_{ij}^r)E_{ij} \leq \sum_{(i,j)\in A^*} (s_i - s_j + \epsilon_i w_i + (1 - x_{ij})S_{ij})$$

$$\iff \sum_{(i,j)\in A^*} (x_{ij}^e + \mu x_{ij}^b + x_{ij}^r)E_{ij} \leq \sum_{(i,j)\in \delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{i \in N\cup R'} s_i^* \tag{37}$$

$$\iff \sum_{(i,j)\in A} (x_{ij}^e + \mu x_{ij}^b + x_{ij}^r)E_{ij} \leq \sum_{(i,j)\in \delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{i \in N\cup R'} s_i^*, \tag{38}$$

where the second transformation is due to the fact $x_{ij}^e = 0, x_{ij}^b = 0, x_{ij}^r = 0, \forall (i,j) \in A \backslash A^*$.

Similarly, we can develop the following valid inequalities for PHEVRDP-D:

$$s_i^* + (1 - x_{ij})S_{ij} \geq s_i + \epsilon_i w_i - \sum_{l\in L_{ij}} (x_{ijl}^e + \mu x_{ijl}^b + x_{ijl}^r)E_{ijl} \qquad \forall (i,j) \in \delta_{n+1}^- \tag{39}$$

$$\sum_{(i,j)\in A} \sum_{l\in L_{ij}} (x_{ijl}^e + \mu x_{ijl}^b + x_{ijl}^r)E_{ijl} \leq \sum_{(i,j)\in \delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{i \in N\cup R'} s_i^*. \tag{40}$$

We refer to the valid inequalities (36)–(40) as *electricity cover inequalities* (ECIs), as they ensure that the electricity consumed by all vehicles is covered by the initial battery charge and recharged electricity.

3.2 Lower bound inequalities

This section will establish a set of valid inequalities by constructing a lower bound for the proposed models.

3.2.1 PHEVRDP-C

By relaxing $\sum_{(i,j)\in A} x_{ij}^f E_{ij}$, $\sum_{(i,j)\in A} x_{ij}^e E_{ij}$, and $\sum_{(i,j)\in A} x_{ij}^b E_{ij}$ to three continuous variables z^f, z^e, and z^b, we can get a lower bound on the objective function (4):

$$\min_{z^f, z^e, z^b} \quad c_f z^f + c_e z^e + c_b z^b + c_e \sum_{(i,j)\in A} x_{ij}^r E_{ij} \tag{41}$$

$$\text{s.t.} \quad z^f + z^e + z^b \geq \sum_{(i,j)\in A} (x_{ij}^f + x_{ij}^b + x_{ij}^e)E_{ij} \tag{42}$$

$$z^e + \mu z^b \leq \sum_{(i,j)\in\delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{(i,j)\in A} x_{ij}^r E_{ij} - \sum_{i\in N\cup R'} s_i^* \tag{43}$$

$$z^f \geq 0, z^e \geq 0, z^b \geq 0, \tag{44}$$

where constraint (43) stems from ECI (38).

Instead of solving model (41)–(44) analytically and using a piecewise function with binary variables to combine multiple objective values together as in Wu et al. (2024), we adopt a simpler and more efficient strategy. This new approach generates a set of valid inequalities in a more compact form, significantly speeding up computations without introducing any additional binary variables. Numerical comparisons indicate that this set of valid inequalities reduces computation time by approximately 20 percent compared to those in Wu et al. (2024). Specifically, by introducing two dual variables, ψ_1 and ψ_2, we can formulate the dual problem for the model (41)–(44) as follows:

$$\max_{\psi_1,\psi_2} \quad \sum_{(i,j)\in A} (x_{ij}^f + x_{ij}^b + x_{ij}^e) E_{ij} \psi_1 - \left(\sum_{(i,j)\in\delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{(i,j)\in A} x_{ij}^r E_{ij} - \sum_{i\in N\cup R'} s_i^* \right) \psi_2 + c_e \sum_{(i,j)\in A} x_{ij}^r E_{ij} \tag{45}$$

$$\text{s.t.} \quad \psi_1 \leq c_f \tag{46}$$

$$\psi_1 - \psi_2 \leq c_e \tag{47}$$

$$\psi_1 - \mu\psi_2 \leq c_b \tag{48}$$

$$\psi_1 \geq 0, \psi_2 \geq 0. \tag{49}$$

When $c_f \leq \frac{c_b - \mu c_e}{1-\mu}$, the feasible area defined by constraints (46)–(49) contains three extreme points: $(0,0)$, $(c_e, 0)$, and $(c_f, c_f - c_e)$. If $c_f \geq \frac{c_b - \mu c_e}{1-\mu}$, it has the extreme points $(0,0)$, $(c_e, 0)$, $(c_f, \frac{c_f - c_b}{\mu})$, and $(\frac{c_b - \mu c_e}{1-\mu}, \frac{c_b - c_e}{1-\mu})$. By using the fact that a linear programming model has an optimal solution on an extreme point of its feasible area, we can establish the following proposition:

Proposition 1. *If $(1-\mu)c_f \leq c_b - \mu c_e$, the following inequalities are valid for PHEVRDP-C:*

$$\sum_{(i,j)\in A} \left(c_f x_{ij}^f + c_e x_{ij}^e + c_b x_{ij}^b \right) E_{ij} \geq c_f \sum_{(i,j)\in A} (x_{ij}^f + x_{ij}^b + x_{ij}^e) E_{ij} - (c_f - c_e) \left(\sum_{(i,j)\in\delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{(i,j)\in A} x_{ij}^r E_{ij} - \sum_{i\in N\cup R'} s_i^* \right). \tag{50}$$

If $(1-\mu)c_f \geq c_b - \mu c_e$, the following inequalities are valid for PHEVRDP-C:

$$\sum_{(i,j)\in A} \left(c_f x_{ij}^f + c_e x_{ij}^e + c_b x_{ij}^b \right) E_{ij} \geq c_f \sum_{(i,j)\in A} (x_{ij}^f + x_{ij}^b + x_{ij}^e) E_{ij} - \frac{c_f - c_b}{\mu} \left(\sum_{(i,j)\in\delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{(i,j)\in A} x_{ij}^r E_{ij} - \sum_{i\in N\cup R'} s_i^* \right) \tag{51}$$

$$\sum_{(i,j)\in A} \left(c_f x_{ij}^f + c_e x_{ij}^e + c_b x_{ij}^b \right) E_{ij} \geq \frac{c_b - \mu c_e}{1-\mu} \sum_{(i,j)\in A} (x_{ij}^f + x_{ij}^b + x_{ij}^e) E_{ij} - \frac{c_b - c_e}{1-\mu} \left(\sum_{(i,j)\in\delta_0^+} s_0 x_{ij} + \sum_{i\in R'} \epsilon_i w_i - \sum_{(i,j)\in A} x_{ij}^r E_{ij} - \sum_{i\in N\cup R'} s_i^* \right). \tag{52}$$

3.2.2 PHEVRDP-D

Similar to Proposition 1, and following the same analytical approach, we can derive the following proposition:

Proposition 2. *If $(1-\mu)c_f \leq c_b - \mu c_e$, the following inequalities are valid for PHEVRDP-D:*

$$\sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(c_f x^f_{ijl} + c_e x^e_{ijl} + c_b x^b_{ijl}\right)E_{ijl} \geq c_f \sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(x^f_{ijl} + x^e_{ijl} + x^b_{ijl}\right)E_{ijl} - (c_f - c_e)\left(\sum_{(i,j)\in\delta^+_0} s_0 x_{ij} + \sum_{i\in R'}\epsilon_i w_i - \sum_{(i,j)\in A}\sum_{l\in L_{ij}} x^r_{ijl}E_{ijl} - \sum_{i\in N\cup R'} s^*_i\right). \tag{53}$$

If $(1-\mu)c_f \geq c_b - \mu c_e$, the following inequalities are valid for PHEVRDP-D:

$$\sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(c_f x^f_{ijl} + c_e x^e_{ijl} + c_b x^b_{ijl}\right)E_{ijl} \geq c_f \sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(x^f_{ijl} + x^e_{ijl} + x^b_{ijl}\right)E_{ijl} - \frac{c_f - c_b}{\mu}\left(\sum_{(i,j)\in\delta^+_0} s_0 x_{ij} + \sum_{i\in R'}\epsilon_i w_i - \sum_{(i,j)\in A}\sum_{l\in L_{ij}} x^r_{ijl}E_{ijl} - \sum_{i\in N\cup R'} s^*_i\right) \tag{54}$$

$$\sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(c_f x^f_{ijl} + c_e x^e_{ijl} + c_b x^b_{ijl}\right)E_{ijl} \geq \frac{c_b - \mu c_e}{1-\mu} \sum_{(i,j)\in A}\sum_{l\in L_{ij}}\left(x^f_{ijl} + x^e_{ijl} + x^b_{ijl}\right)E_{ijl} - \frac{c_b - c_e}{1-\mu}\left(\sum_{(i,j)\in\delta^+_0} s_0 x_{ij} + \sum_{i\in R'}\epsilon_i w_i - \sum_{(i,j)\in A}\sum_{l\in L_{ij}} x^r_{ijl}E_{ijl} - \sum_{i\in N\cup R'} s^*_i\right). \tag{55}$$

We call the valid inequalities (50)–(52) and (53)–(55) *lower bound inequalities* (LBIs), because they are developed from the lower bounds of the proposed models.

3.3 Recharging inequalities

The presence of recharging stations adds significant complexity to the problem. We have observed that the following simple *recharging inequalities* (RIs),

$$\sum_{(i,j)\in\delta^+_l} x_{ij}\overline{B} \geq \epsilon_l w_l + s_l \qquad \forall l \in R', \tag{56}$$

can significantly accelerate the computations when combined with ECIs and LBIs, as will be shown in Section 5. For example, when used together, they can optimally solve all instances with 25 or 30 customers and three recharging stations in under one minute on average. In contrast, the model with only ECIs and LBIs fails to solve some instances with 25 or 30 customers and three recharging stations to optimality within two hours of computing time.

3.4 Symmetry breaking inequalities for the duplicated recharging stations

As mentioned in Section 2.2, we use dummy nodes for recharging stations to allow multiple visits to each station. For a recharging station duplicated κ times in the graph, there will be $\kappa+1$ nodes representing this station. This approach introduces two symmetry issues. First, if the station is visited κ_c times, there are $\binom{\kappa+1}{\kappa_c}$ ways to select the corresponding nodes. Second, there are $\binom{\kappa_c}{2}$ ways to exchange the visiting sequence among the nodes of the station.

Drawing on the work of Montoya et al. (2017), which dealt with symmetry in the electric VRP, we can eliminate the above symmetries by establishing a visiting order for each recharging station and its corresponding dummy nodes. If nodes i and j represent a recharging station and its dummy nodes, we can address the first symmetry issue by using the following valid inequalities to ensure that node i can be visited only if node j is also visited:

$$\sum_{(i,k)\in\delta_i^+} x_{ik} \geq \sum_{(j,l)\in\delta_j^+} x_{jl}. \tag{57}$$

To break the second symmetry issue, we can use the following valid inequalities to require that node i should be visited earlier than node j:

$$t_i \leq t_j + \left(1 - \sum_{(j,l)\in\delta_j^+} x_{jl}\right)T, \tag{58}$$

where T is a large value that can be set as the maximum duration of the trips for all vehicles. Constraint (58) is only valid if node j is visited; otherwise, it is invalid. In the following sections, we will refer to constraints (57)–(58) as *symmetry breaking inequalities* (SBIs).

4 Solution method

This section describes the solution methods for solving the PHEVRDP-C, which include a linearization technique, an exact algorithm, and a heuristic algorithm. Note that all methods can be applied to the PHEVRDP-D, with the exception of the linearization process.

4.1 Linearization

The non-linear terms v_{ij}^2 and $1/v_{ij}$ make the problem intractable. To address this, we apply a linearization framework similar to that of Wu et al. (2024), adapting it to suit our problem, as detailed in Appendix B. This process linearizes the PHEVRDP-C, converting it into an MILP.

4.2 Branch-and-cut algorithm

Branch-and-cut is a popular exact method for solving routing problems (Bard et al. 2002, Lysgaard et al. 2004). This section introduces the key aspects of the algorithm used to solve the PHEVRDP-C.

4.2.1 Preprocessing

When the time windows are tight, it is possible to eliminate certain binary variables by identifying incompatible arcs (Bard et al. 2002). In our problem setting, the vehicle speed on each arc can be adjusted within a specified range, and the minimum travel time on each arc can be calculated by dividing the distance by the maximum speed. If the minimum travel time exceeds the maximum allowable travel time on arc (i, j), namely $\overline{\tau}_j - \underline{\tau}_i < d_{ij}/\overline{v}_{ij}$, then it is considered infeasible and we can safely remove it from the graph.

For each arc, we can calculate the minimum energy consumption by assuming that the vehicle is running at the lower speed bound. If the energy consumption exceeds the battery capacity of the PHEV, then the electricity mode can also be infeasible. Similarly, if the minimum energy consumption, multiplied by the coefficient of the electricity energy split in the boost mode, exceeds the battery capacity of a PHEV, then the boost mode is infeasible.

4.2.2 Separation problem

For the separation of subtour elimination constraints (24)–(25), we utilize the CVRPSEP package developed by Lysgaard and Reinelt (2003). If an integer solution at a branch-and-bound node violates the subtour elimination constraints, we incorporate the violated constraints into the model and re-solve the problem at the current node. This iterative process continues until no violated constraints remain.

In the course of implementation, we have observed that adding subtour elimination constraints to fractional solutions can help speed up calculations. However, adding too many of them can also slow down the algorithm. To obtain a good trade-off, we limit the number of such constraints by adding them only when they are violated by at least 1 percent.

4.2.3 Subgradient cut

The linearization process introduces a class of subgradient cuts (see Equation (99) in Appendix B). When an integer solution is found in the branch-and-bound tree, we first verify if it complies with the subtour elimination constraints. Subsequently, we assess whether it satisfies the subgradient cut. If the solution violates this cut, it is incorporated into the model. During implementation, we found that this verification process is crucial to avoid reintroducing the same subgradient cut into the model.

4.3 Iterated optimization heuristic

To solve the problem heuristically, we employ an Iterated Optimization (IO) heuristic comprising two main stages. The first stage, referred to as *RouteOptimization*, calculates the optimal route to minimize total energy consumption. The second stage, referred to as *Mode&SpeedOptimization*, determines the optimal running speed and driving mode for each arc. The details are outlined below.

RouteOptimization This step determines the optimal routes for a fleet of vehicles with the objective of minimizing total energy consumption. We proceed under the assumption of a fixed speed for each arc, allowing us to calculate both travel time and energy consumption on each arc. Then, we can formulate a VRPTW aimed at minimizing the overall energy consumption of all vehicles, while considering time windows and capacity constraints. Let $\hat{V}$ be the node set, $\hat{A}$ be the arc set, and $\hat{v}_{ij}$ and $\hat{E}_{ij}$ be the given speed and energy consumption on each arc, respectively. Finally, the VRPTW can be formulated as follows:

$$\min_{x_{ij}} \quad \sum_{(i,j)\in\hat{A}} x_{ij}\hat{E}_{ij} \tag{59}$$

$$\text{s.t.} \quad t_i + w_i + \frac{d_{ij}}{\hat{v}_{ij}} - t_j \le (1 - x_{ij})T_{ij} \qquad \forall (i,j)\in\hat{A} \tag{60}$$

$$\sum_{(i,j)\in\delta^+(\Omega)} x_{ij} \ge \Delta(\Omega) \qquad \forall \Omega\subseteq\hat{V}, \Omega\neq\emptyset \tag{61}$$

$$\underline{\tau}_i \le t_i \le \overline{\tau}_i \qquad \forall i\in\hat{V} \tag{62}$$

$$x_{ij}\in\{0,1\} \qquad \forall (i,j)\in\hat{A} \tag{63}$$

(17)–(19), (21).

The model above can be solved heuristically by off-the-shelf optimizers, such as the routing library of OR-Tools (Furnon and Perron 2024).

Mode&SpeedOptimization In this step, we determine the driving mode and optimal speed on each arc using the simplified PHEVRDP-C with 'fixed routes'. For scenarios without a recharging station, the 'fixed routes' can be directly given by the results of the *RouteOptimization* stage. However, for scenarios with recharging stations, the 'fixed routes' can also be derived from the results of the *RouteOptimization* stage, excluding the arcs connected to the recharging stations. In this case, the $Mode\&SpeedOptimization$ stage will also include the selection of recharging stations. It is worth noting that our proposed valid inequalities can also be applied to the simplified models, allowing for their efficient solution through the branch-and-cut algorithm.

Referring to the method proposed by Demir et al. (2012) for solving a pollution-routing problem, these stages are executed iteratively, with further details provided in Appendix C due to limited space. Importantly, the IO heuristic can serve as an initial solution for the branch-and-cut algorithm or be used directly.

5 Computational experiments

This section presents the findings from computational experiments carried out to evaluate the effectiveness of the proposed models and solution methods. The tests were run on an AMD Rome 7532 2.40 GHz CPU with a 256M L3 cache. The branch-and-cut algorithm and IO heuristic are implemented in Python, utilizing Gurobi Optimizer 10.0.3 as the solver, with a 7,200-second time limit per computation. The instances and codes are available at the URL: `https://github.com/fuliang93/PHEVRDP.git`.

5.1 Instances

Since the PHEVRDP is a new problem, there are currently no benchmark instances for it. To evaluate the proposed models and solution methods, we have adapted the Solomon (1987) VRPTW benchmark instances (group C) to create our own instances.

Each instance is denoted by α_β_π, where α, β, and π represent the instance index, the number of charging stations, and the number of customers, respectively. For instance α_β_π, we select the first π nodes from the Solomon instance α to serve as customers, and assign the following β nodes as recharging stations. The units of time windows and distance are set as minutes and kilometers. The lower bound of speed on each arc is set to 30 km/h, while the upper bound of speed is randomly determined between 50 km/h and 80 km/h. For each node, we randomly generate the elevation, ranging from 0 to 200 meters, which is then utilized to calculate the slopes of the road between the nodes. Drivetrain efficiency and regeneration efficiency are set to be 0.45 and 0.15, respectively, which are similar to the values used in Yi and Bauer (2018). Note that using different values will not affect the computational efficiency of the proposed algorithm. To set the fleet size for each instance, we first identify the minimum number of vehicles needed for the viability of the proposed PHEVRDP-C, then allocate one additional vehicle beyond this minimum.

In our instances, each PHEV is equipped with a battery capacity of 14.4 kilowatt-hours, which corresponds to the battery size of a 2022 Ford Escape PHEV (Latham 2022). The unit costs of energy consumption in fuel-only, electric-only, boost, and energy recuperation modes are set to 1, 0.5, 0.7, and -0.5, respectively. In addition, the coefficient of the electricity energy split in the boost mode is set to 0.5.

5.2 Performance of the solution method

This section evaluates the effectiveness of the valid inequalities introduced in Section 3 and the solution method outlined in Section 4.

5.2.1 Instances without a recharging station

First, we evaluate the effectiveness of the proposed valid inequalities, and the computational results for PHEVRDP-C are summarized in Table 3. It can be seen that the original PHEVRDP-C cannot be solved to optimality for some instances with 25 customers and for the majority with 30 customers. As a result, we have chosen not to solve the instances with more customers. However, the PHEVRDP-C with LBIs or ECIs can be solved to optimality much more quickly. The PHEVRDP-C with only LBIs can be solved to optimality for all instances with 40 or fewer customers, and for most instances with 50 customers, except for one with a small optimality gap (less than 1%). The PHEVRDP-C with only ECIs can be solved to optimality for all instances with 50 or fewer customers. From the results, it is evident that ECIs are marginally more effective than LBIs.

When generating the initial solution with the IO heuristic, we set n_i as the number of nodes plus 5 times the number of vehicles, and $n_j = 1$. The results are presented in the INI column in Table 3. When the number of customers is small, the PHEVRDP-C with LBIs, ECIs, and initial solution has a longer calculation time than the PHEVRDP-C with LBIs or ECIs because of the calculation time taken for the initial solution. However, as the number of customers increases, the PHEVRDP-C with LBIs, ECIs, and initial solution can be solved more quickly, as the time allocated for the initial solution becomes marginal in the context of the overall computation time.

As mentioned before, the IO heuristic can directly function as a solution method. We set n_i as the number of nodes plus 5 times the number of vehicles. Through testing the algorithm with different values of n_j, we have found that $n_j = 6$ shows good balance between calculation time and accuracy. Although the calculation time for instances with a small number of customers is longer than the PHEVRDP-C with ECIs, LBIs, and initial solution, the calculation time for instances with a large number of customers is much shorter. This indicates that the IO heuristic is able to produce high-quality solutions within a reasonable calculation time for large-scale instances.

Table 3: Performance of the solution methods for PHEVRDP-C on instances without a recharging station

		Ori			LBI			ECI			ECI+LBI		
#Cus	*#Ins*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*
15	17	17	66.8	0.00	17	2.2	0.00	17	**1.6**	0.00	17	1.8	0.00
25	17	14	1938.2	3.32	17	11.1	0.00	17	**9.7**	0.00	17	10.4	0.00
30	17	7	4961.1	10.22	17	27.0	0.00	17	**20.4**	0.00	17	21.0	0.00
40	17				17	140.6	0.01	17	98.0	0.00	17	96.8	0.00
50	17				16	653.3	0.07	17	532.7	0.00	17	510.3	0.00
60	17				14	1587.2	0.56	14	1549.2	0.59	14	1885.1	**0.55**
70	17				13	2110.5	0.85	14	2336.3	0.82	14	1833.0	**0.66**

		ECI+LBI+INI			INI			IO heuristic		
#Cus	*#Ins*	*#Opt*	*aTime*	*aGap*	*aDif*	*aTime*	*#Iter*	*aDif*	*aTime*	*#Iter*
15	17	17	3.7	0.00	5.79	2.5	1.1	0.59	14.2	10.9
25	17	17	11.3	0.00	6.46	5.4	1.2	0.00	32.3	12.2
30	17	17	24.1	0.00	5.44	8.7	1.3	0.32	60.4	15.2
40	17	17	**92.7**	0.00	11.60	13.7	1.3	1.53	89.7	14.5
50	17	17	**462.3**	0.00	12.41	20.9	1.6	3.27	117.5	14.1
60	17	14	1488.8	**0.55**	10.35	25.7	1.4	2.25	180.8	16.5
70	17	14	1862.0	0.85	10.14	32.4	1.5	2.84	201.4	15.5

Ori: the original model with the subtour elimination constraints; ECI: Ori with ECIs; LBI: Ori with LBIs; ECI+LBI: Ori with ECIs and LBIs; INI: the initial solution calculated by the IO heuristic; ECI+LBI+INI: Ori with ECIs, LBIs, and INI; $\#Cus$: the number of customers; $\#Ins$: the number of instances; $\#Opt$: the number of instances that are solved to optimality within 2 hours; $aTime$: the average computation time (s); $aGap$: average optimality gap of the instances (%); $aDif$: the average difference in objectives between the heuristic solution and the optimal solution (%); $\#Iter$: the average iteration number of the IO heuristic. The bold indicates the minimum average calculation time when all instances are solved to optimality, and the minimum average gap when some instances are not solved to optimality.

Second, we solve the PHEVRDP-D with ECIs, LBIs, and initial solution on a set of instances. The computational results are summarized in Table 4. Here, the notations 5 km/h and 10 km/h indicate that the continuous speed has been discretized to distinct speed levels at intervals of 5 km/h and 10 km/h, respectively. Note that we have also incorporated the minimum and maximum bounds of the speed into the set of speed levels, in cases where these bounds were not already represented by the previously discretized speed levels.

Table 4: Performance of PHEVRDP-D on instances without a recharging station

		5 km/h				10 km/h			
$\#Cus$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$aDif^*$	$\#Opt$	$aTime$	$aGap$	$aDif^*$
15	17	17	3.7	0.00	0.02	17	3.5	0.00	0.07
25	17	17	10.1	0.00	0.06	17	8.8	0.00	0.25
30	17	17	20.6	0.00	0.05	17	18.9	0.00	0.22
40	17	17	101.3	0.00	0.06	17	63.5	0.00	0.24
50	17	17	240.1	0.00	0.08	17	180.5	0.00	0.23
60	17	15	1355.7	3.17	0.13	15	1136.6	2.50	0.21
70	17	14	1711.4	2.67	0.24	14	1530.6	2.86	0.25

$aDif^*$: the average difference in objectives between PHEVRDP-D and PHEVRDP-C (%).

It can be seen that all instances with 50 or fewer customers have been optimally solved for PHEVRDP-D. When comparing the two discretization intervals, PHEVRDP-D with a 10 km/h interval is solved more quickly than with a 5 km/h interval. This increased efficiency is attributed to the reduced number of discretization levels, which consequently decreases the number of binary variables. For all instances, the objective values obtained by PHEVRDP-D are slightly higher than those obtained by PHEVRDP-C, with the difference being negligible.

5.2.2 Instances with recharging stations

In this section, we first assume that each recharging station can be visited at most once. The computational results from evaluating the proposed inequalities on PHEVRDP-C are detailed in Table 5. It can be seen that the inclusion of recharging options significantly increases computational complexity, as the original model requires considerably more computation time compared to its counterpart without a recharging station.

While LBIs and ECIs can significantly reduce the calculation time, they fall short of solving certain instances with 25 customers and two recharging stations, and they are unable to solve the majority of instances with 30 customers and three recharging stations. RIs on their own do not significantly enhance computational efficiency. Consequently, we refrained from performing separate evaluations of PHEVRDP-C with LBIs, ECIs, RIs, and the combination of LBIs and ECIs for instances with 40 or more customers. However, when RIs are integrated with ECIs and LBIs, they significantly enhance computational performance. In addition, the initial solution can help accelerate the computation. Using the IO heuristic with parameters consistent with those detailed in the preceding section, we observe that it is capable of producing high-quality solutions within a reasonable computation time for large-scale instances.

An interesting observation is the decrease in objective values compared to instances without a recharging station, as shown in the column $aRat^-$ of Table 5. This indicates that the presence of recharging stations can lead to a significant cost reduction. Therefore, it is beneficial to consider recharging opportunities when devising routing plans for PHEVs.

Furthermore, we solve the PHEVRDP-D with ECIs, LBIs, RIs, and initial solution across a diverse range of instances, as presented in Table 6. Consistent with the results for instances without a recharging station, solving the PHEVRDP-D with a 10 km/h interval is faster than with a 5 km/h interval. Additionally, the objective values achieved by PHEVRDP-D are marginally higher than those obtained by PHEVRDP-C.

Finally, we test the methods on instances with duplicated recharging stations, allowing each station to be visited multiple times. Computational results are presented in Table 8 in Appendix E, where 'Total number' denotes the maximum number of allowable visits for each recharging station. It is evident that allowing multiple visits to each recharging station significantly increases the complexity of the PHEVRDP-C. This is reflected in the substantial increase in computation time for PHEVRDP-C with ECIs, LBIs, and RIs, especially when the 'Total number' is set to 3. However, the proposed SBIs offer a significant improvement by not only reducing computation time but also solving more instances to optimality. While the reduction in the objective value is not substantial compared to instances without duplicated recharging stations, the computational burden becomes much heavier. Therefore, in the following numerical studies, we focus on the instances without duplicated recharging stations.

Table 5: Performance of the solution methods for PHEVRDP-C in the instances with recharging stations

			Ori			LBI			ECI		
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$
15	2	17	17	202.4	0.00	17	15.8	0.00	17	17.2	0.00
25	3	17	12	3280.5	3.58	15	1451.7	0.61	15	1225.0	0.51
30	3	17	7	4633.4	8.57	12	3201.7	2.80	13	2478.2	2.20

			RI			ECI+LBI			ECI+LBI+RI		
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$
15	2	17	17	223.7	0.00	17	16.1	0.00	17	**3.9**	0.00
25	3	17	12	3204.4	3.58	15	1244.0	0.39	17	44.0	0.00
30	3	17	7	4615.3	8.80	12	2669.6	2.44	17	40.9	0.00
40	4	17							16	640.1	**0.17**
50	5	17							13	2254.0	1.35
60	6	17							12	2798.8	**1.59**

			ECI+LBI+RI+INI				INI			IO heuristic		
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$aRat^-$	$aDif$	$aTime$	$\#Iter$	$aDif$	$aTime$	$\#Iter$
15	2	17	17	6.2	0.00	12.70	7.86	3.3	1.4	2.61	14.9	10.1
25	3	17	17	**35.1**	0.00	4.81	7.74	7.1	1.4	1.19	40.5	13.3
30	3	17	17	**40.0**	0.00	3.10	6.97	10.8	1.5	1.01	59.9	13.5
40	4	17	16	585.5	0.20	4.84	7.03	19.3	1.8	1.64	105.8	14.4
50	5	17	14	1857.4	**1.31**	5.93	5.55	32.8	2.3	2.89	146.9	13.7
60	6	17	13	2524.8	1.69	3.62	7.35	42.6	1.9	2.36	251.4	16.9

RI: Ori with RIs; ECI+LBI+RI: Ori with ECIs, LBIs, and RIs; ECI+LBI+RI+INI: Ori with ECIs, LBIs, RIs, and INI; $\#Cha$: the number of recharging stations; $aRat^-$: the average rate of reduction in the objective value when considering the inclusion of recharging stations, as compared to cases that do not incorporate recharging stations (%).

Table 6: Performance of PHEVRDP-D on instances with recharging stations

			5 km/h				10 km/h			
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$aDif^*$	$\#Opt$	$aTime$	$aGap$	$aDif^*$
15	2	17	17	8.5	0.00	0.03	17	6.6	0.00	0.10
25	3	17	17	42.8	0.00	0.09	17	31.6	0.00	0.13
30	3	17	17	35.5	0.00	0.05	17	26.7	0.00	0.22
40	4	17	17	462.8	0.00	0.06	17	373.5	0.00	0.23
50	5	17	14	1631.0	0.56	0.31	15	1420.6	0.56	0.23
60	6	17	14	2112.8	0.78	0.45	14	1817.0	0.77	0.46

5.3 The formulation with a vehicle index

This section tests the PHEVRDP-C with a vehicle index, as detailed in Appendix A, which is a straightforward extension of the TSP model proposed in Wu et al. (2024). The computational results are presented in Table 7, along with those for the formulation without a vehicle index for easier comparison.

It is clear that solving the PHEVRDP-C with a vehicle index is more challenging than its counterpart without a vehicle index, as the computation time for the original model increases significantly, and a larger number of instances fail to reach optimality. The introduction of ECIs, LBIs, and RIs significantly reduces computation time. For example, they save approximately 99 percent of computing time for instances with 15 customers and no recharging stations, and over 98 percent for instances with 15 customers and two recharging stations. However, even with these improvements, the PHEVRDP-C with a vehicle index remains more computationally intensive compared to the formulation without a vehicle index. For example, the average computation time for instances with 30 customers and no recharging stations is over 25 times that of the formulation without a vehicle index. Additionally, the model with a vehicle index fails to optimally solve certain instances with 25 or 30 customers and three recharging stations within two hours of computing time, whereas the version without a vehicle index can solve all these instances to optimality quickly, with an average calculation time of under one minute. The PHEVRDP-D with a vehicle index yields similar results, but we omit them here due to space constraints.

Table 7: Performance of the PHEVRDP-C with a vehicle index

			Formulation with a vehicle index						Formulation without a vehicle index					
			Ori			Valid inequalities			Ori			Valid inequalities		
#Cus	*#Cha*	*#Ins*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*	*#Opt*	*aTime*	*aGap*
15	0	17	17	256.2	0.00	17	2.6	0.00	17	66.8	0.00	17	**1.8**	0.00
25	0	17	9	3735.3	6.37	17	27.3	0.00	14	1938.2	3.32	17	**10.4**	0.00
30	0	17	6	5059.8	15.71	17	529.4	0.00	7	4961.1	10.22	17	**21.0**	0.00
15	2	17	16	883.2	0.32	17	17.4	0.00	17	202.4	0.00	17	**3.9**	0.00
25	3	17	5	5236.2	8.50	14	1568.8	0.88	12	3280.5	3.58	17	**44.0**	0.00
30	3	17	9	4172.6	9.22	13	2167.4	2.38	7	4633.4	8.57	17	**40.9**	0.00

Formulation with a vehicle index: the PHEVRDP-C with a vehicle index. Formulation without a vehicle index: the PHEVRDP-C model proposed in this paper. Ori: the original model. Valid inequalities: Ori with ECIs and LBIs for instances without recharging stations, Ori with ECIs, LBIs, and RIs for instances with recharging stations.

5.4 The value of speed optimization

A natural question that arises is why speed optimization is included in routing the PHEV fleet. To address this question, we have created a series of instances that operate with a fixed speed, enabling us to evaluate the impact of speed optimization on the energy cost of the PHEV fleet. It is worth noting that the PHEVRDP with fixed speeds represents a special case of the PHEVRDP-C. Consequently, the valid inequalities introduced for the PHEVRDP-C, along with their associated solution methodologies, are also applicable for solving the PHEVRDP with fixed speeds. In this section, we incorporate the valid inequalities and utilize the branch-and-cut algorithm to solve the problem.

Table 9 in Appendix E shows the computational results of the PHEVRDP with fixed speeds, where 40 km/h, 50 km/h, and speed limit indicate that the vehicle is traveling at those respective speeds on each arc. It can be seen that the PHEVRDP with fixed speeds can be efficiently solved to optimality using our proposed solution methods, even for large instances. However, the PHEVRDP with fixed speeds yields a much higher cost compared to PHEVRDP-C, by nearly 15 percent at 40 km/h, 30 percent at 50 km/h, and 60 percent when using speed limits. This underscores the significance of incorporating speed optimization in the PHEV fleet routing process.

5.5 Sensitivity analysis

In this section, we conduct a sensitivity analysis on the proposed PHEVRDP-C to assess the effects of varying the number of vehicles and the cost of drivers. We restrict our analysis to instances with 30 or fewer customers. As established in the previous section, the initial solution provides negligible computational benefit for instances of this size. Consequently, we apply only the valid inequalities and employ the branch-and-cut algorithm to solve the instances under consideration.

Table 10 in Appendix E shows the results of PHEVRDP-C with varying vehicle numbers. 'Minimum' represents the smallest number of vehicles required for the feasibility of PHEVRDP-C, while 'Minimum + 1' denotes the number of vehicles used in other numerical studies. Solving the PHEVRDP-C with the minimum number is more challenging, possibly due to instances where the minimum number is equal to 1, resulting in the VRP becoming a TSP. However, in this case, we utilize the separation algorithm for subtour elimination inequalities for the VRP. In terms of objective values, the PHEVRDP-C with a larger number of vehicles will yield an average decrease of nearly 10 percent compared to the PHEVRDP-C with the minimum number of vehicles, as the feasible region of the model expands with more vehicles.

To assess the impact of driver wages on the PHEVRDP-C, we modified the objective function as described in Appendix D. We then conducted tests on the PHEVRDP-C using objective function (110) with varying driver costs per unit time, and the numerical results are presented in Table 11 in Appendix E. It can be seen that integrating driver costs into the model results in longer computational times, because of the increased complexity of the objective function. An interesting observation is that the increase in objective function values is more pronounced when recharging options are considered, for instance, exceeding four times the average increase observed in cases without recharging stations when the unit cost of the driver is set at 50. This phenomenon can be attributed to driver costs influencing the operational strategy, resulting in faster speeds on each arc and a reduction in the utilization of recharging opportunities or recharging time, both of which will increase energy consumption costs.

6 Conclusion

In this study, we have introduced a PHEVRDP, which simultaneously optimizes speeds, routes, and operating modes to minimize the energy consumption cost for a fleet of PHEVs. The PHEVRDP with continuous speed variables is formulated as an MINLP model, while the version with discretized speed variables is formulated as an MILP model. To efficiently solve these models, this paper has introduced valid inequalities to strengthen them, subsequently employing a branch-and-cut algorithm and an IO heuristic. The computational experiments have demonstrated that the proposed methods can optimally or heuristically solve instances with a realistic number of customers within a reasonable time, making them suitable for addressing daily tour planning problems. Moreover, our proposed models and solution methods can also be effectively applied to the fixed-speed VRPTW for PHEVs, achieving optimal solutions with high efficiency. The experiments have shown that integrating speed decisions into the VRPTW for PHEVs can effectively reduce the energy consumption cost, underscoring the significance of incorporating speed optimization into routing planning problems for PHEVs. Furthermore, the numerical experiments have revealed that considering recharging possibilities and increasing the number of vehicles when routing a fleet of PHEVs can contribute to reducing the energy consumption costs. As driver expenses are usually dependent on the travel time, which may necessitate the vehicle to complete the journey more quickly, this could lead to higher energy consumption costs. This effect becomes more significant when recharging possibilities are taken into account.

This paper suggests potential areas for future research. First, given that vehicles on roads are subject to uncertain traffic speeds, it would be valuable to incorporate traffic speed uncertainty into the proposed models. Second, traffic signals may introduce stop-and-go conditions, increasing both energy consumption and travel time. Integrating the constraints and effects of these signals into our PHEVRDP would further improve the model's realism and practical value.

Appendices

Appendix A PHEVRDP-C with a vehicle index

For the decision variables, let x_{ijk} be a binary variable taking value 1 if and only if vehicle k is traveling on arc (i,j), and x^f_{ijk} be a binary variable taking value 1 if and only if vehicle k is operating in fuel mode on arc (i,j). Similarly, x^e_{ijk}, x^b_{ijk}, and x^r_{ijk} take value 1 if and only if vehicle k is operating in electric, boost, or energy recuperation mode on arc (i,j), respectively. Let v_{ijk} be the speed of vehicle k on arc (i,j), s_{ik} be the state of charge of vehicle k at node i, t_{ik} be the arrival time of vehicle k at node i, w_{ik} be the recharging time of vehicle k at recharging station i, and E_{ijk} be the energy demand of vehicle k on arc (i,j) depending on the speed v_{ijk}. Then the PHEVRDP with a vehicle index can be formulated as an MINLP model as follows:

$$\min \sum_{k\in K}\sum_{(i,j)\in A}\left(c_f x^f_{ijk} + c_e x^e_{ijk} + c_b x^b_{ijk} + c_e x^r_{ijk}\right) E_{ijk} \tag{64}$$

$$\text{s.t.} \sum_{k\in K}\sum_{(i,j)\in\delta_i^+} x_{ijk} = 1 \qquad \forall i\in N \tag{65}$$

$$\sum_{k\in K}\sum_{(i,j)\in\delta_i^+} x_{ijk} \le 1 \qquad \forall i\in R' \tag{66}$$

$$\sum_{(i,j)\in\delta_i^+} x_{ijk} - \sum_{(j,i)\in\delta_i^-} x_{jik} = 0 \qquad \forall i\in N\cup R', k\in K \tag{67}$$

$$\sum_{(i,j)\in\delta_0^+} x_{ijk} = 1 \qquad \forall k\in K \tag{68}$$

$$\sum_{(i,j)\in\delta_{n+1}^-} x_{ijk} = 1 \qquad \forall k\in K \tag{69}$$

$$x^f_{ijk} + x^e_{ijk} + x^b_{ijk} + x^r_{ijk} = x_{ijk} \qquad \forall (i,j)\in A, k\in K \tag{70}$$

$$\begin{aligned} E_{ijk} = {} & \eta_g d_{ij} x_{ijk}\left(mg\sin\theta_{ij} + \frac{1}{2}C_d\rho F v_{ijk}^2 + C_r mg\cos\theta_{ij}\right) \\ & + \left(\frac{1}{\eta_d} - \eta_g\right) d_{ij} x_{ijk} \max\left\{mg\sin\theta_{ij} + \frac{1}{2}C_d\rho F v_{ijk}^2 + C_r mg\cos\theta_{ij}, 0\right\} \qquad \forall (i,j)\in A, k\in K \end{aligned} \tag{71}$$

$$(x^f_{ijk} + x^e_{ijk} + x^b_{ijk} - 1)M_{ij} \le E_{ijk} \qquad \forall (i,j)\in A, k\in K \tag{72}$$

$$(1 - x^r_{ijk})M_{ij} \ge E_{ijk} \qquad \forall (i,j)\in A, k\in K \tag{73}$$

$$s_{ik} + \epsilon_i w_{ik} - (x^e_{ijk} + \mu x^b_{ijk} + x^r_{ijk})E_{ijk} \ge s_{jk} - (1 - x_{ijk})S_{ij} \qquad \forall (i,j)\in A, k\in K \tag{74}$$

$$s_{ik} + \epsilon_i w_{ik} - (x^e_{ijk} + \mu x^b_{ijk} + x^r_{ijk})E_{ijk} \le s_{jk} + (1 - x_{ijk})S_{ij} \qquad \forall (i,j)\in A, k\in K \tag{75}$$

$$t_{ik} + w_{ik} + \frac{d_{ij}}{v_{ijk}} - t_{jk} \le (1 - x_{ijk})T_{ij} \qquad \forall (i,j)\in A, k\in K \tag{76}$$

$$\underline{B} \le s_{ik} \le \overline{B} \qquad \forall i\in V', k\in K \tag{77}$$

$$\underline{\tau}_i \le t_{ik} \le \overline{\tau}_i \qquad \forall i\in N\cup\{n+1\}, k\in K \tag{78}$$

$$w_{ik} \ge 0 \qquad \forall i\in V', k\in K \tag{79}$$

$$\underline{v}_{ij} \le v_{ijk} \le \overline{v}_{ij} \qquad \forall (i,j)\in A, k\in K \tag{80}$$

$$\sum_{k\in K}\sum_{(i,j)\in\delta^+(\Omega)} x_{ijk} \ge \Delta(\Omega) \qquad \forall \Omega\subseteq N\cup R', \Omega\neq\emptyset \tag{81}$$

$$x_{ijk}, x^f_{ijk}, x^e_{ijk}, x^b_{ijk}, x^r_{ijk} \in \{0,1\} \qquad \forall (i,j)\in A, k\in K. \tag{82}$$

The objective function (64) minimizes the total energy consumption cost of all vehicles, E_{ijk} will be negative in energy recuperation mode. Constraints (65) ensure that each customer is visited by exactly one vehicle, guaranteeing that all demands are satisfied. Constraints (66) require that each visit to a recharging station is used at most once (note that not all vertices in set R' must be visited). Constraints (67)–(69) define a source-to-sink path for each vehicle. Constraints (70) ensure that the vehicle k can only run in one mode on each arc. Constraints (71) calculate the energy consumption of each vehicle on each arc, which will be 0 if arc (i,j) is not in the optimal route of vehicle k. Constraints (72) require that fuel-only, electric-only, and boost modes are not chosen under a negative energy consumption. Constraints (73) enforce that the energy recuperation mode cannot be chosen under a positive energy consumption. Constraints (74)–(75) track the battery charge level of each vehicle at each node. Constraints (76) determine the arrival time of each vehicle at each node. Constraints (77) impose limits on each vehicle's battery charge level, constraining it within the minimum and maximum charge levels. Constraints (78) are time windows. Constraints (79) ensure that the recharging time at each node is non-negative. Constraints (80) limit the speeds of each vehicle across the entire network. In this formulation, we assume that $\underline{v} > 0$ to ensure the feasibility of constraints (68). Constraints (81) enforce capacity limits for each vehicle and eliminate subtours.

To reduce the number of decision variables, we can consolidate the speed variables $v_{ijk}, k \in K$ and energy consumption variables $E_{ijk}, k \in K$ on each arc into two variables: v_{ij} and E_{ij}. These variables represent the vehicle speed and energy consumption on arc (i,j), respectively. This consolidation is feasible because each arc can only be visited by a vehicle once, meaning that each arc has a unique speed and energy consumption value. As a result, model (64)–(82) can be reformulated as follows:

$$\min \quad \sum_{k\in K}\sum_{(i,j)\in A}\left(c_f x^f_{ijk} + c_e x^e_{ijk} + c_b x^b_{ijk} + c_e x^r_{ijk}\right) E_{ij} \tag{83}$$

$$\text{s.t.} \quad E_{ij} = \eta_g d_{ij}\left(mg\sin\theta_{ij} + \frac{1}{2}C_d\rho F v_{ij}^2 + C_r mg\cos\theta_{ij}\right)\sum_{k\in K} x_{ijk}$$
$$+\left(\frac{1}{\eta_d} - \eta_g\right) d_{ij}\max\left\{mg\sin\theta_{ij} + \frac{1}{2}C_d\rho F v_{ij}^2 + C_r mg\cos\theta_{ij}, 0\right\}\sum_{k\in K} x_{ijk} \quad \forall (i,j)\in A \tag{84}$$

$$\left(\sum_{k\in K}\left(x^f_{ijk} + x^e_{ijk} + x^b_{ijk}\right) - 1\right) M_{ij} \le E_{ij} \quad \forall (i,j)\in A \tag{85}$$

$$\left(1 - \sum_{k\in K} x^r_{ijk}\right) M_{ij} \ge E_{ij} \quad \forall (i,j)\in A \tag{86}$$

$$s_{ik} + \epsilon_i w_{ik} - \left(x^e_{ijk} + \mu x^b_{ijk} + x^r_{ijk}\right) E_{ij} \ge s_{jk} - (1 - x_{ijk}) S_{ij} \quad \forall (i,j)\in A, k\in K \tag{87}$$

$$s_{ik} + \epsilon_i w_{ik} - \left(x^e_{ijk} + \mu x^b_{ijk} + x^r_{ijk}\right) E_{ij} \le s_{jk} + (1 - x_{ijk}) S_{ij} \quad \forall (i,j)\in A, k\in K \tag{88}$$

$$t_{ik} + w_{ik} + \frac{d_{ij}}{v_{ij}} - t_{jk} \le (1 - x_{ijk}) T_{ij} \quad \forall (i,j)\in A, k\in K \tag{89}$$

$$\underline{v}_{ij} \le v_{ij} \le \overline{v}_{ij} \quad \forall (i,j)\in A \tag{90}$$

(65)–(70), (77)–(79), (81)–(82),

where constraints (84) require that the energy consumption E_{ij} must be 0 if arc (i,j) is not part of the optimal routes.

To address this problem, we can develop electricity cover inequalities, lower bound inequalities, and recharging inequalities similar to those outlined in Section 3. Additionally, we can employ the solution methods described in Section 4 to solve this problem. Furthermore, we have found that implementing the following symmetry-breaking inequalities can enhance the speed of the calculation.

A.1 Symmetry breaking inequalities for the formulation with a vehicle index

There are two main symmetry issues that stem from $|K|$ homogeneous vehicles (Adulyasak et al. 2014). First, there is the fact that there are $\binom{|K|}{|K_d|}$ possibilities for dispatching $|K_d|$ vehicles. Second, there are $|K_d|!$ choices to swap the routes that are assigned to each dispatched vehicle. These two symmetry issues can slow down the branch-and-bound process because of the duplication of nodes (Sherali and Smith 2001).

To break the first symmetry issue, we can use the following valid inequalities to require that vehicle $k+1$ can be dispatched only if vehicle k is also dispatched:

$$\sum_{(i,j)\in\delta_0^+} x_{ijk} \geq \sum_{(i,j)\in\delta_0^+} x_{ij,k+1} \qquad \forall k \in \{1,2,...,|K|-1\}. \tag{91}$$

As for the second symmetry issue, we can use the following lexicographic ordering constraints:

$$\sum_{o=1}^{p} 2^{p-o} \sum_{(i,j)\in\delta_o^+} x_{ijk} \geq \sum_{o=1}^{p} 2^{p-o} \sum_{(i,j)\in\delta_o^+} x_{ij,k+1} \qquad \forall p \in N \cup R', k \in \{1,2,...,|K|-1\}. \tag{92}$$

Appendix B Linearization

This section provides a detailed explanation of the linearization process for the non-linear PHEVRDP-C. We first introduce new variables $u_{ij} = v_{ij}^2, (i,j) \in A$. Then, constraints (6), (10), and (22) can be reformulated as follows:

$$E_{ij} \geq \frac{d_{ij}}{\eta_d}\left(mg\sin\theta_{ij} + \frac{1}{2}C_d\rho A u_{ij} + C_r mg\cos\theta_{ij}\right)x_{ij} \qquad \forall (i,j) \in A \tag{93}$$

$$E_{ij} \geq \eta_g d_{ij}\left(mg\sin\theta_{ij} + \frac{1}{2}C_d\rho A u_{ij} + C_r mg\cos\theta_{ij}\right)x_{ij} \qquad \forall (i,j) \in A \tag{94}$$

$$\underline{v}_{ij}^2 \leq u_{ij} \leq \overline{v}_{ij}^2 \qquad \forall (i,j) \in A \tag{95}$$

$$t_i + w_i + \frac{d_{ij}}{\sqrt{u_{ij}}} - t_j \leq (1-x_{ij})T_{ij} \qquad \forall (i,j) \in A. \tag{96}$$

Constraints (96) are non-linear because of the term $1/\sqrt{u_{ij}}$, which is a convex function in u_{ij}. We can derive a subgradient cut as follows:

- **First**, by substituting term $1/\sqrt{u_{ij}}$ with a new continuous variable q_{ij}, constraints (96) are reformulated to the following constraints:

$$t_i + w_i + d_{ij}q_{ij} - t_j \leq (1-x_{ij})T_{ij} \qquad \forall (i,j) \in A \tag{97}$$

$$q_{ij} \geq 0 \qquad \forall (i,j) \in A. \tag{98}$$

- **Second**, the tangent line of function $1/\sqrt{u_{ij}}$ at point $(\hat{u}_{ij}, \hat{u}_{ij}^{-\frac{1}{2}})$ is $-\frac{1}{2}\hat{u}_{ij}^{-\frac{3}{2}}(u_{ij} - \hat{u}_{ij}) + \hat{u}_{ij}^{-\frac{1}{2}}$. Then the subgradient cut for term $u_{ij}^{-\frac{1}{2}}$ can be derived as follows:

$$q_{ij} \geq -\frac{1}{2}\hat{u}_{ij}^{-\frac{3}{2}}(u_{ij} - \hat{u}_{ij}x_{ij}) + \hat{u}_{ij}^{-\frac{1}{2}}x_{ij} \qquad \forall (i,j) \in A. \tag{99}$$

  If $x_{ij} = 0$, the right-hand side of constraints will be negative and the cut is inactive; otherwise, the cut is added to the problem.

Additionally, constraints (93)–(95) can be linearized into the following constraints:

$$E_{ij} \geq \frac{d_{ij}}{\eta_d}\left(mg\sin\theta_{ij}x_{ij} + \frac{1}{2}C_d\rho A u_{ij} + C_r mg\cos\theta_{ij}x_{ij}\right) \qquad \forall (i,j) \in A \tag{100}$$

$$E_{ij} \geq \eta_g d_{ij}\left(mg\sin\theta_{ij}x_{ij} + \frac{1}{2}C_d\rho A u_{ij} + C_r mg\cos\theta_{ij}x_{ij}\right) \qquad \forall (i,j) \in A \tag{101}$$

$$\underline{v}_{ij}^2 x_{ij} \leq u_{ij} \leq \overline{v}_{ij}^2 x_{ij} \qquad \forall (i,j) \in A. \tag{102}$$

To linearize the non-linear terms $x^f_{ij}E_{ij}$, $x^e_{ij}E_{ij}$, $x^b_{ij}E_{ij}$, and $x^r_{ij}E_{ij}$ in the model, we introduce new continuous variables ω^f_{ij}, ω^e_{ij}, ω^b_{ij}, and ω^r_{ij} subject to the following constraints:

$$\omega^f_{ij} + \omega^e_{ij} + \omega^b_{ij} + \omega^r_{ij} \geq E_{ij} \qquad \forall (i,j) \in A \qquad (103)$$
$$\omega^f_{ij} \leq x^f_{ij} M_{ij} \qquad \forall (i,j) \in A \qquad (104)$$
$$\omega^e_{ij} \leq x^e_{ij} M_{ij} \qquad \forall (i,j) \in A \qquad (105)$$
$$\omega^b_{ij} \leq x^b_{ij} M_{ij} \qquad \forall (i,j) \in A \qquad (106)$$
$$\omega^r_{ij} \geq -x^r_{ij} M_{ij} \qquad \forall (i,j) \in A \qquad (107)$$
$$\omega^f_{ij} \geq 0, \omega^e_{ij} \geq 0, \omega^b_{ij} \geq 0, \omega^r_{ij} \leq 0 \qquad \forall (i,j) \in A. \qquad (108)$$

Appendix C Iterated optimization heuristic

This section introduces the implementation details of the Iterated Optimization (IO) heuristic. Let S represent the solution, which includes the speeds, operating modes, and routes. The speed matrix on all arcs is denoted by $\mathbf{v}$, and the routes are denoted by $\mathbf{r}$. Additionally, n_i represents the maximum iteration number, and n_j represents the maximum number of iterations during which the solution is not improved. The outline of the iterated optimization algorithm is described in Algorithm 1. Here, $InitializeSpeedMatrix(\mathbf{v}_{\max})$ refers to the initialization of the speed matrix with the speed limits. $ReinitializeSpeedMatrix(\mathbf{v}_{\min})$ involves reinitializing the speed matrix with the lower bounds of speed. Additionally, $UpdateSpeedMatrix(S_{\text{Best}})$ denotes the update of the speed matrix based on the best solution S_{Best}. In $Perturbation(\mathbf{v}_{\text{Best}})$, we randomly set the speed between the best speed, upper and lower bounds of speed on each arc of the longest route of the best solution, and randomly set the upper and lower bounds of speed on the arcs in addition to the best routes.

Algorithm 1 Iterated Optimization Heuristic

1. $S_{\text{Best}} \leftarrow \emptyset$; $Obj(S_{\text{Best}}) \leftarrow +\infty$; /* S_{Best} = Best Solution
2. $(i, j) \leftarrow (0, 0)$;
3. $\mathbf{v} \leftarrow$ InitializeSpeedMatrix($\mathbf{v}_{\max}$);
4. **while** $i < n_i$:
   (a) $i \leftarrow i + 1$;
   (b) $\mathbf{r} \leftarrow$ RouteOptimization($\mathbf{v}$) ;
   (c) $S \leftarrow$ Mode&SpeedOptimization($\mathbf{r}$) ;
   (d) **while** $j < n_j$:
      i. **if** $Obj(S) < Obj(S_{\text{Best}})$:
         A. $j \leftarrow 0$;
         B. $Obj(S_{\text{Best}}) \leftarrow Obj(S)$;
         C. $\mathbf{v} \leftarrow$ UpdateSpeedMatrix(S);
      ii. **elif** $Obj(S) \geq Obj(S_{\text{Best}})$:
         A. $j \leftarrow j + 1$;
         B. $\mathbf{v} \leftarrow$ Perturbation($\mathbf{v}_{\text{Best}}$) ;
      iii. **if** $j = n'_j$: ReinitializeSpeedMatrix($\mathbf{v}_{\min}$) ;
   (e) **if** $j \geq n_j$: Break.

Return: S_{Best}.

The details of the IO heuristic are shown in Algorithm 1. We begin by setting the initial values of the speed matrix to the upper bounds for speed. Following this initialization, we enter the primary iterative process. Within each iteration, we calculate routes that are optimized for energy efficiency, taking into account the current state of the speed matrix. Leveraging these optimized routes, we then identify the most favorable driving modes and the corresponding driving speeds. Should the newly computed objective value prove to be lower than the previously recorded best objective value, we proceed to update both the best objective value and the speed matrix accordingly. In the event that the objective value fails to surpass the best objective value, we apply a perturbation to the speed matrix. The iterative process is terminated if there is no improvement in the objective value after n_j consecutive iterations, at which point we output the current best result. Conversely, if improvements are observed, we continue the iterations, ultimately reporting the best solution obtained over the course of n_i iterations. An interesting observation during the implementation phase revealed that by reinitializing the speed matrix to the lower speed bounds when no improvement in the objective value is noted over n'_j iterations — a count that is less than n_j — we are potentially able to find a better solution.

The IO heuristic can directly serve as a heuristic for the PHEVRDP-C. The selection of n_i and n_j is crucial as it allows for intentional calibration between solution precision and computational time. When generating an initial solution, it is recommended to use small values of n_i and n_j to minimize the computational burden. Conversely, opting for large values of n_i and n_j can lead to a more accurate heuristic solution, but it also comes with a higher computational cost. This trade-off increases the likelihood of obtaining a superior solution while incurring a higher computational cost.

Appendix D The objective function accounting for driver cost

Drivers are typically payed based on travel time, which includes the running time on each arc, recharging time at each recharging station, and the waiting time at each node. To calculate the travel time of each driver, we introduce new variables $t_i^*, i \in N \cup R'$ subject to the following:

$$t_i + w_i + \frac{d_{ij}}{v_{i,n+1}} - t_i^* \leq (1 - x_{i,n+1})T_{i,n+1} \qquad \forall i \in N \cup R', \tag{109}$$

where t_i^* denotes the arrival time at the ending depot when arc $(i, n+1)$ is part of the solution, and is set to 0 if arc $(i, n+1)$ is not included in the solution.

Let c_d be the cost per unit time for the drivers, then the objective function (4) will be modified to the follows:

$$\min \sum_{(i,j)\in A} \left(c_f x_{ij}^f + c_e x_{ij}^e + c_b x_{ij}^b + c_e x_{ij}^r \right) E_{ij} + c_d \sum_{i \in N \cup R'} t_i^*. \tag{110}$$

In this function, a trade-off exists between driver cost and energy consumption cost: as driver costs increase, vehicles tend to operate at faster speeds, leading to higher energy consumption costs.

Appendix E Computational results of the numerical studies

This section presents some computational results from the numerical studies in Section 5. Table 8 shows the numerical results for the PHEVRDP-C using the instances with duplicated recharging stations. Table 9 shows the numerical results for the PHEVRDP with fixed speeds. Table 10 shows the numerical results of the PHEVRDP-C with varying numbers of vehicles. Finally, Table 11 shows the results of the PHEVRDP-C considering different driver costs.

Table 8: Performance of the PHEVRDP-C on instances with duplicated recharging stations

			Total number = 2								Total number = 3							
			ECI+LBI+RI			ECI+LBI+RI+SBI					ECI+LBI+RI			ECI+LBI+RI+SBI			
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$	$aRat^-$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$	$aRat^-$
15	2	17	17	11.7	0.00	17	8.5	0.00	0.76	17	432.7	0.00	17	16.0	0.00	0.76
25	3	17	17	262.0	0.00	17	189.9	0.00	0.00	15	1256.6	0.36	17	404.2	0.00	0.00
30	3	17	17	39.1	0.00	17	35.9	0.00	0.00	17	54.3	0.00	17	43.6	0.00	0.00
40	4	17	15	1295.1	0.53	16	781.5	0.38	0.36	14	1643.9	1.09	16	1018.2	0.61	0.34
50	5	17	6	5029.8	4.74	7	4661.6	4.22		1	6953.4	9.20	1	6817.2	8.69	

Total number: the count of the duplicated nodes plus the original node for a recharging station; ECI+LBI+RI+SBI: Ori with ECIs, LBIs, RIs, and SBIs; $aRat^-$: the average rate of decrease in the objective value when duplicating recharging stations, compared to cases without duplication; the empty cells indicate that $aRat^-$ is not applicable, as most instances have not achieved optimal solutions

Table 9: Performance of the PHEVRDP with fixed speeds

			40 km/h				50 km/h				Speed limit			
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$aRat^+$	$\#Opt$	$aTime$	$aGap$	$aRat^+$	$\#Opt$	$aTime$	$aGap$	$aRat^+$
15	0	17	17	0.6	0.00	13.51	17	0.6	0.00	32.43	17	0.7	0.00	63.48
25	0	17	17	3.8	0.00	17.51	17	4.2	0.00	30.87	17	3.0	0.00	59.34
30	0	17	17	6.3	0.00	15.86	17	7.5	0.00	28.45	17	4.1	0.00	55.07
40	0	17	17	23.4	0.00	15.44	17	21.3	0.00	28.54	17	17.6	0.00	58.73
50	0	17	17	73.3	0.00	14.33	17	100.2	0.00	26.48	17	26.3	0.00	53.47
60	0	17	15	1079.1	0.33	14.99	14	1354.0	0.34	27.33	15	957.3	0.17	56.10
70	0	17	14	1379.7	0.39	14.13	14	1362.6	0.36	26.75	14	1357.3	0.19	54.62
15	2	17	17	2.2	0.00	16.53	17	2.3	0.00	37.35	17	2.1	0.00	67.70
25	3	17	17	25.7	0.00	16.94	17	17.4	0.00	30.94	17	12.0	0.00	60.22
30	3	17	17	21.9	0.00	16.35	17	23.3	0.00	29.78	17	11.1	0.00	58.32
40	4	17	17	174.6	0.00	15.37	17	110.5	0.00	28.53	17	52.6	0.00	58.95
50	5	17	14	1642.2	0.64	15.42	16	1072.8	0.13	28.69	17	141.3	0.00	57.52
60	6	17	12	2350.9	1.07	15.19	14	1770.8	0.68	28.16	14	1551.8	0.37	55.56

$aRat^+$: the average rate of increase in the objective value when using a fixed speed on each arc, as compared to cases incorporating speed optimization (%).

Table 10: Performance of the PHEVRDP-C with varying numbers of vehicles

			Minimum			Minimum+1				Minimum+3			
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$\#Opt$	$aTime$	$aGap$	$aRat^-$	$\#Opt$	$aTime$	$aGap$	$aRat^-$
15	0	17	17	1.2	0.00	17	1.8	0.00	11.63	17	1.3	0.00	11.63
25	0	17	17	14.7	0.00	17	10.4	0.00	8.93	17	11.3	0.00	8.93
30	0	17	16	453.5	1.18	17	21.0	0.00	6.47	17	21.0	0.00	6.54
15	2	17	17	4.9	0.00	17	3.9	0.00	13.48	17	3.9	0.00	13.48
25	3	17	17	99.6	0.00	17	44.0	0.00	8.40	17	40.7	0.00	8.40
30	3	17	15	967.0	3.01	17	40.9	0.00	7.22	17	38.7	0.00	7.28

$aRat^-$: the average rate of decrease in the objective value when increasing the number of vehicles, as compared to cases with the minimum number of vehicles (%)

Table 11: Performance of the PHEVRDP-C with different driver costs

			$c_d = 10$				$c_d = 50$				$c_d = 100$			
$\#Cus$	$\#Cha$	$\#Ins$	$\#Opt$	$aTime$	$aGap$	$aRat^+$	$\#Opt$	$aTime$	$aGap$	$aRat^+$	$\#Opt$	$aTime$	$aGap$	$aRat^+$
15	0	17	17	1.1	0.00	0.80	17	1.4	0.00	4.02	17	1.2	0.00	8.01
25	0	17	17	8.7	0.00	0.80	17	9.7	0.00	3.99	17	11.3	0.00	7.99
30	0	17	17	21.1	0.00	0.78	17	29.8	0.00	3.80	17	56.0	0.00	7.47
15	2	17	17	4.7	0.00	18.92	17	5.3	0.00	22.11	17	5.0	0.00	26.10
25	3	17	17	84.3	0.00	12.31	17	137.9	0.00	15.21	17	235.6	0.00	18.84
30	3	17	15	943.8	3.03	12.66	15	1006.4	2.95	15.18	17	985.0	3.05	18.88

$aRat^+$: the average rate of increase in the objective value when considering the driver costs, as compared to cases where the driver costs are not included (%)

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