The maximum independent set problem and augmenting graphs

Alain Hertz
GERAD – École Polytechnique
Montréal, Canada

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Augmenting set of co-authors

Michael Gerber
Vadim V. Lozin
David Schindl

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Independent set : subset of pairwise non-adjacent vertices

Independence number $\alpha(G)$: size of a maximum independent set in G

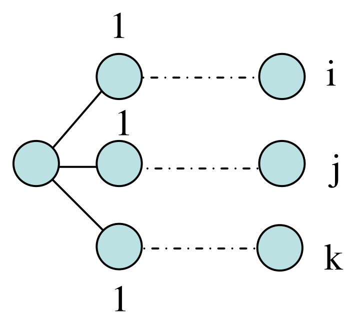
Stable = independent

Problems :

- $\alpha(G)=?$ MAXIMUM INDEPENDENT SET PROBLEM (MIS)
- Given k, determine if G contains an independent set of size k

The MIS is NP-hard

Alekseev has proved that if a graph H has a connected component which is not of the form $S_{i,j,k,}$ then the MIS is NP-hard in the class of H-free graphs



Corollary

The MIS is NP-hard in the class of Triangle-free graphs, Square-free graphs, etc.

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An independent set S is maximal if no other independent set properly contains S

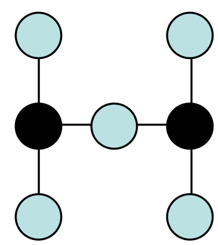
Problem : generate all maximal independent sets

- Tsukiyama, Ide, Ariyoshi and Shirakawa (1977)
- Lawler, Lenstra and Rinnooy Kan (1980)
- Johnson and Yannakakis (1988)

The independent dominating set problem

Find a maximal independent set of minimum cardinality

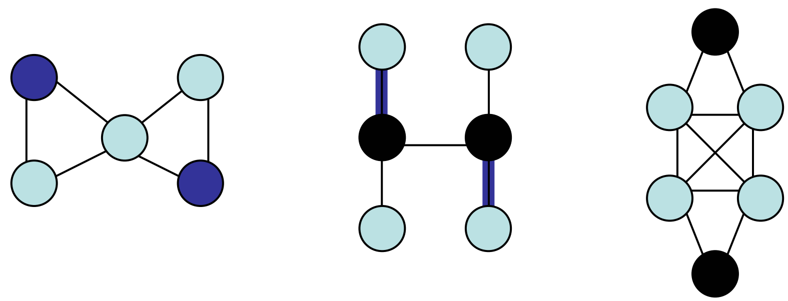
(NP-hard even for bipartite graphs)



S is an independent set in G ⟷ S is a clique in the complement of G

S is an independent set in G ⟷ V-S is a vertex cover in G

M is a matching in G ⟷ M is a stable in the line graph L(G)



Vertex Packing problem

Find an independent set of maximum total weight

Maximum dissociation set problem

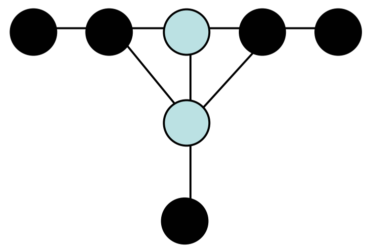
Find a subset of vertices of maximum size
inducing a subgraph with vertex degree at most 1

NP-hard for bipartite graphs

Maximum induced matching problem

Find a subset of vertices of maximum size
inducing a subgraph with vertex degree exactly 1

NP-hard for bipartite graphs with maximum degree 3



Maximum dissociation set : 5

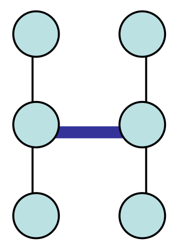
Maximum induced matching : 4 (2 edges)

Maximum matching : 3

Minimum independent edge dominating set problem

Find a maximal matching of minimum cardinality

Yannakakis and Gavril (1980)



In view of the NP-hardness of the MIS

(1) Non-polynomial-time algorithms

(2) Polynomial-time algorithms providing approximate solutions

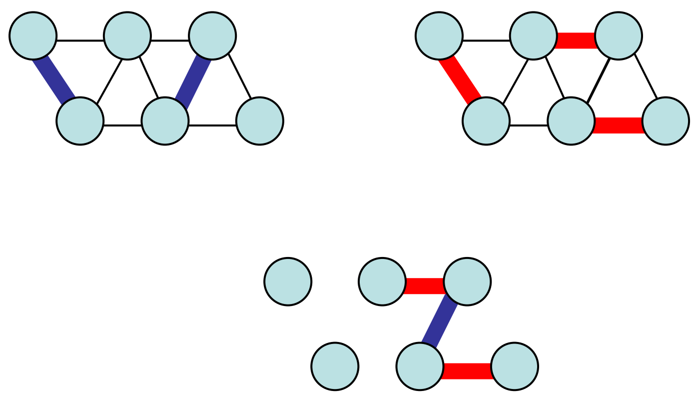
(3) Polynomial-time algorithms that solve the problem exactly for graphs belonging to special classes

Håstad (1999)

non-exact algorithms cannot approximate the size of a maximum independent set within a factor of $n^{1-\varepsilon}$

Theorem

A matching in a graph is maximum if and only if there are no augmenting chains with respect to the matching



Let S be an independent set in a graph G

- The vertices in S are **black**
- The others are **white**

A bipartite graph H=(W,B,E) is **augmenting** for S if

(1) B is a subset of S and W is a subset of V-S

(2) N(W)∩(S-B) is the empty set

(3) W has more vertices than B

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If H=(W,B,E) is augmenting for S then (S-B)∪W is an independent set with more vertices than S.

If S is not of maximum size, then there exists a larger independent set S' and the subgraph induced by (S-S')∪(S'-S) is augmenting for S

Theorem

An independent set S if maximum if and only if there are no augmenting graphs for S.

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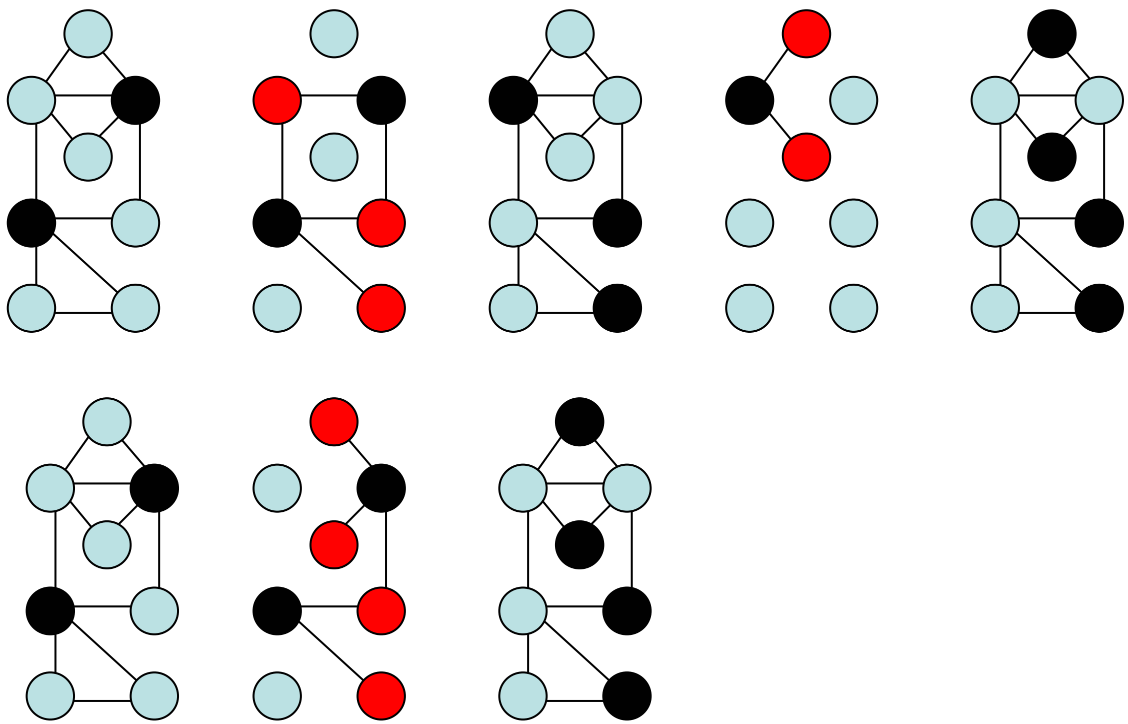
Algorithm

Begin with any independent set S

As long as S admits an augmenting graph H, apply H-augmentation to S

For a polynomial-time algorithm on has to

1. Find a complete list of augmenting graphs in the class under consideration
2. Develop polynomial-time algorithms for detecting all augmenting graphs in the class



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Definition

An augmenting graph for S is **minimal** if no proper induced subgraph of H is augmenting for S

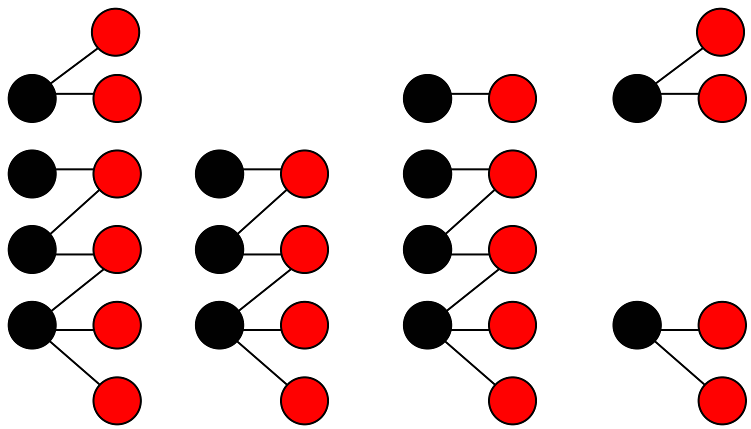
Theorem

If H=(W,B,E) is a minimal augmenting graph for S then

(1) H is connected

(2) $|B|=|W|-1$

(3) For every subset A of B : $|A|<|N_w(A)|$



Characterization of augmenting graph

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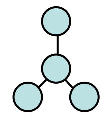
Claw-free graphs ($S_{1,1,1}$)



No bipartite claw-free graph has a vertex of degree more than 2

Corollary

a connected claw-free bipartite graph is an even cycle or a chain

Cycles of even length and chains of odd length are not augmenting

Theorem

Every minimal claw-free augmenting graph is a chain of even length

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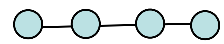
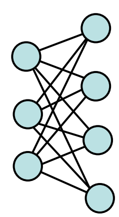
P_4 -free graphs ($S_{0,1,2}$)



Every connected P_4-free graph is complete bipartite

Theorem

Every minimal P_4-free augmenting graph is a $K_{n,n+1}$



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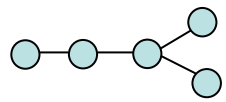
Fork-free graphs ($S_{1,1,2}$)



Let $G=(V_1,V_2,E)$ be a bipartite graph.

The bipartite complement B(G) of G is the bipartite graph $(V_1,V_2,(V_1 \times V_2)-E)$

If G is such that $\Delta(B(G)) < 2$ then G is called a complex

Theorem (Alekseev, 1999)

If G is a connected bipartite fork-free graph then either $\Delta(G)<3$ or $\Delta(B(G))<2$

Corollary (Alekseev, 1999)

Every minimal fork-free augmenting graph is either a chain of even length or a complex

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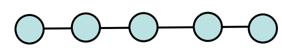
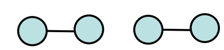
P_5-free graphs ($S_{0,2,2}$)



Theorem

Every connected P_5-free bipartite graph is $2K_2$-free

$2K_2$-free graphs = chain graphs = difference graphs



NO POLYNOMIAL-TIME ALGORITHM IS KNOWN

TO DETECT $2K_2$-FREE BIPARTITE GRAPHS

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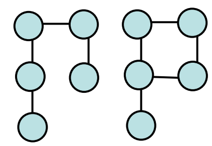
(P_5,banner)-free graphs



The MIS in banner-free graphs : NP-hard
The MIS in P_5-free graphs : complexity open

Theorem (Lozin, 2000)

Every minimal (P_5,banner)-free augmenting graph is a complete bipartite graph

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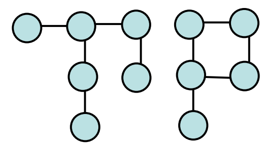
($S_{1,2,2}$,banner)-free graphs



Lemma

A connected bipartite banner-free graph that contains a C_4 is complete bipartite

Lemma

A minimal ($S_{1,2,2}$,C_4)-free augmenting graph is claw-free

Theorem (Hertz, Lozin, 2003)

Every minimal ($S_{1,2,2}$,banner)-free augmenting graph is either complete bipartite or a chain of even length

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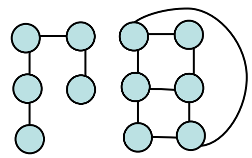
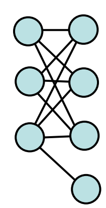
$(P_5, K_{3,3}\text{-}e)$-free graphs

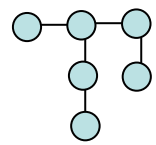


Theorem (Gerber, Hertz, Schindl, 2004)

Every minimal $(P_5, K_{3,3}\text{-}e)$-free augmenting graph is either complete bipartite or a graph obtained from a complete bipartite graph $K_{n,n}$ by adding a single vertex with exactly one neighbor in the opposite part.



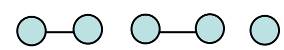
$S_{1,2,2}$-free graphs



A graph is prime is any two distinct vertices have different neighborhoods.

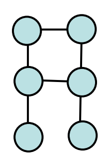
Theorem (Lozin, 2000)

Every prime $S_{1,2,2}$-free bipartite graph is $K_{1,3}$ or $B(P_5)$-free.



Theorem (Hertz, Lozin, 2003)

Every prime $(S_{1,2,2},A)$ -free bipartite graph is $S_{1,1,2}$-free.



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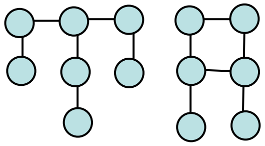
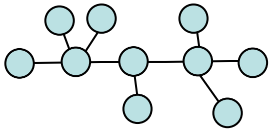
$(S_{2,2,2},A)$ -free graphs

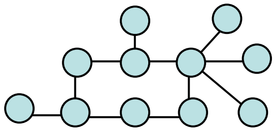


If G is such that $\Delta(B(G)) < 2$ then G is called a complex

A caterpillar : a tree that becomes a path by removing the pendant vertices



A long circular caterpillar : a graph that becomes a C_k (k>4) by removing the pendent vertices



Theorem (Boliac, Lozin, 2001)

Every prime $(S_{2,2,2},A)$ -free bipartite graph is either a caterpillar or a long circular caterpillar or a complex

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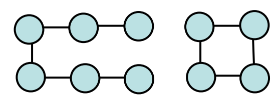
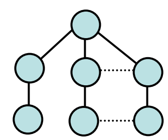
(P_6,C_4) -free graphs



A simple augmenting tree :



Theorem (Mosca, 1999)

Every (P_6,C_4) -free augmenting graph is a simple augmenting tree

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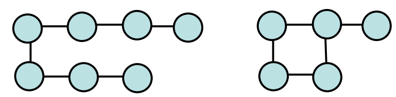
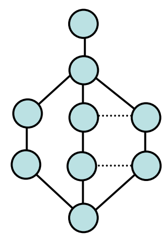
(P_7,banner) -free graphs



An augmenting plant :



Theorem (Alekseev,Lozin, 2000)

Every minimal (P_7,banner) -free augmenting graph is either complete bipartite or a simple augmenting tree or an augmenting plant

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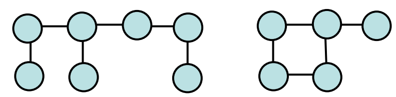
$(S_{1,2,3}$,banner) -free graphs



Theorem (Alekseev,Lozin, 2000)

Every minimal $(S_{1,2,3}$,banner) -free augmenting graph is either complete bipartite or a simple augmenting tree or an augmenting plant or a chain of even length.

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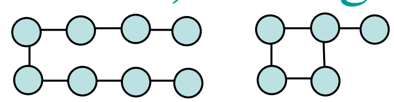
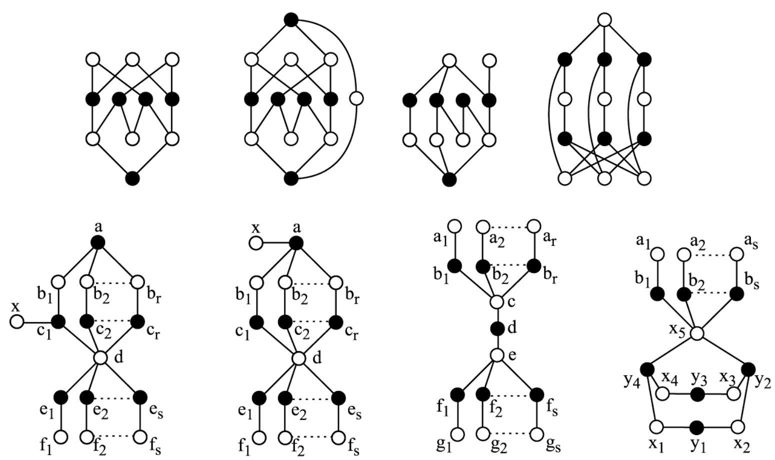
(P_8,banner) -free graphs

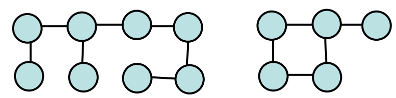


Theorem (Gerber, Hertz, Lozin, 2004)

Every minimal (P_8,banner) -free augmenting graph is either a complete bipartite $K_{n,n+1}$ or one of the following graphs

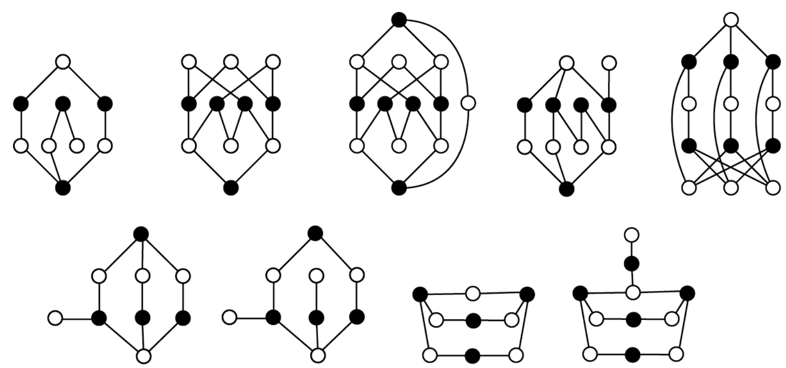


$(S_{1,2,4}$,banner) -free graphs



Theorem (Gerber, Hertz, Lozin, 2004)

Every minimal $(S_{1,2,4}$,banner) -free augmenting graph is either a complete bipartite graph or chain of even length, or a simple augmenting tree, or an augmenting plant or one of the following graphs.



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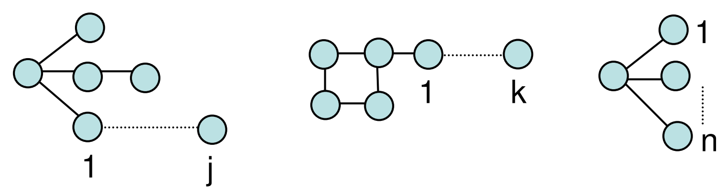
$(S_{1,2,j}$, banner$_k$, $K_{1,n})$ -free graphs



Theorem (Gerber, Hertz, Lozin, 2004)

Let j, k and n be any three integers

The class of $(S_{1,2,j}$, banner$_k$, $K_{1,n})$ -free graphs contains finitely many minimal augmenting graphs different from chains.

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Finding
Augmenting graphs

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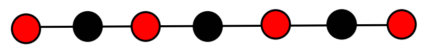
Augmenting Chains



Polynomial algorithms

- Claw-free graphs : Minty and Sbihi (1980)
- $S_{1,2,3}$-free graphs Gerber, Hertz, Lozin (2003)
- ($S_{1,2,i}$, banner)-free graphs : Hertz, Lozin, Schindl (2003)

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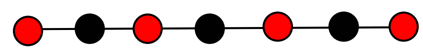
Augmenting Chains in claw-free graphs
Minty and Sbihi (1980)



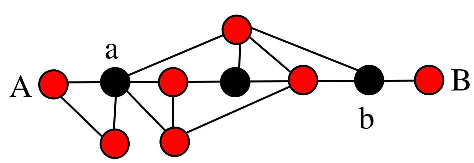
Let A and B be two non-adjacent white vertices, each of which has exactly one black neighbor. Let a and b denote their black neighbor.

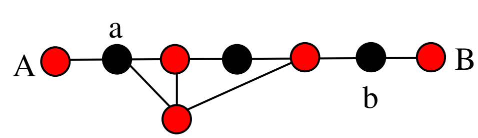
We look for an augmenting chain connecting A to B

If a=b then (A,a=b,B) is an augmenting chain.

Hence, we can assume that a and b are distinct vertices

We can also assume that any white vertex different from A and B is not adjacent to A and B and has exactly two black neighbors (other white vertices cannot occur in an augmenting chain connecting A to B.





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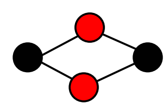
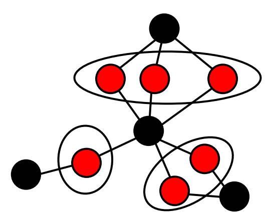
Two white vertices having the same black neighbors are called similar

The similarity is an equivalence relation and an augmenting chain contains at most one vertex in each class of similarity



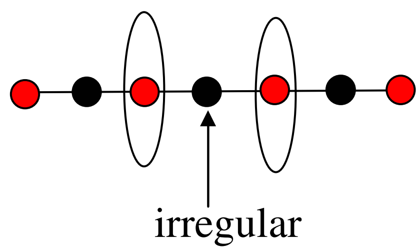
The similarity classes in the neighborhood of a black vertex are called wings



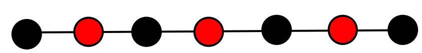
A black vertex different from a and b and with more than two wings is said regular; otherwise it is irregular.

R={regular black vertices} ∪ {a,b}

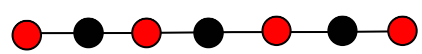
Remark :



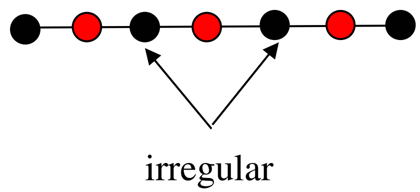
A black alternating chain (with possible chords linking white vertices)



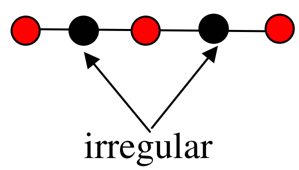
A white alternating chain (with possible chords linking white vertices)



An irregular black alternating chain = IBAP = chordless black alternating chain in which all black vertices except the termini are irregular



An irregular white alternating chain = IWAP = white alternating chain obtained by removing the termini of an IBAP



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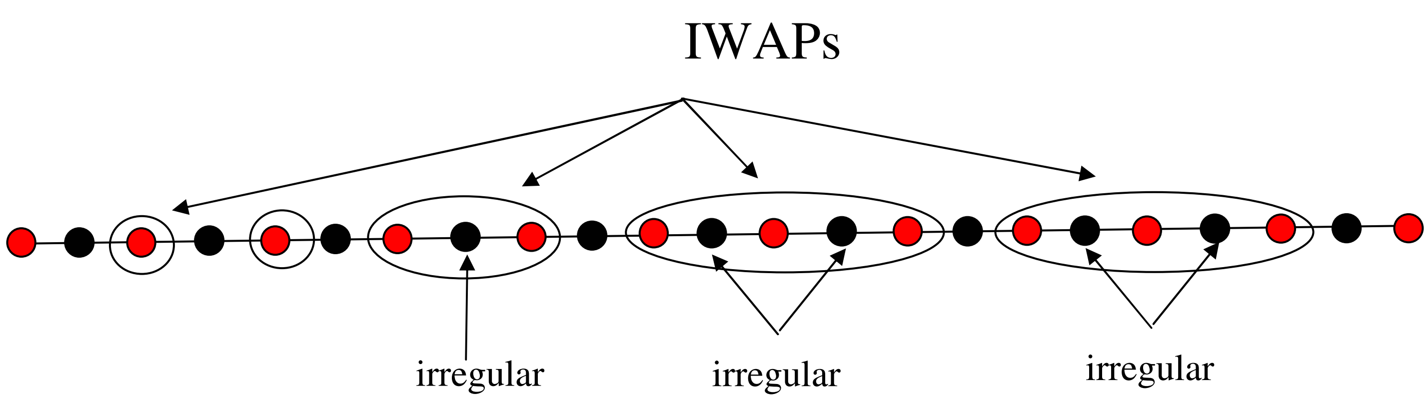
An augmenting chain



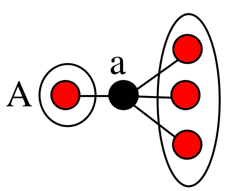
Decomposition of the neighborhood of each black vertex v

into two subsets $N_1(v)$ and $N_2(v)$

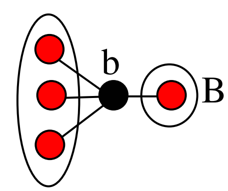
called node classes of v

in such a way that no two vertices in the same node class can occur in the chain augmenting chain

$N_1(a)=\{A\}, N_2(a)=N(a)-\{A\}$

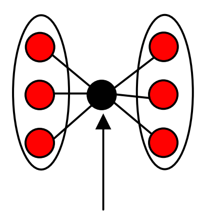


$N_1(b)=\{B\}, N_2(b)=N(b)-\{B\}$



If v is irregular then the node classes are the wings

called node classes of v

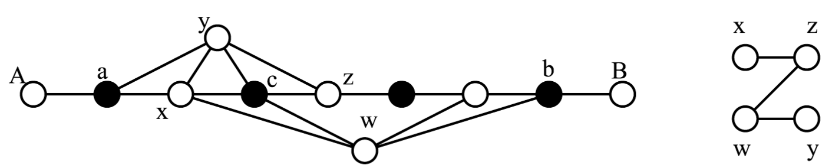


If v is regular then

- adjacent white neighbors of v are in the same node class
- similar white neighbors of v are in the same node class

Consider the graph H(v) where

- The vertex set of H(v) is N(v)
- Two vertices u and w in H(v) are linked by an edge if and only if they are non-similar and non-adjacent



H(c)

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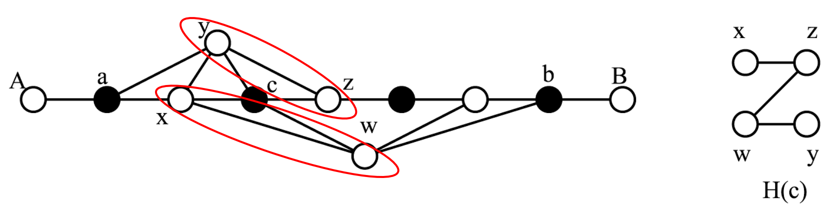
THEOREM (Minty 1980)

H(v) is bipartite for all regular vertices v

The node classes of a regular vertex v are the two parts of H(v)



Let u and v be two white vertices.

Are u and v the endpoints of an IWAP ?

Condition : u and v must have a black neighbor in R

So let b_0 be a black vertex in R and let W_1 be one wing of w

Minty has shown how to determine the set of pairs (u,v) such that u belongs to W_1 and there exists an IWAP with termini u and v

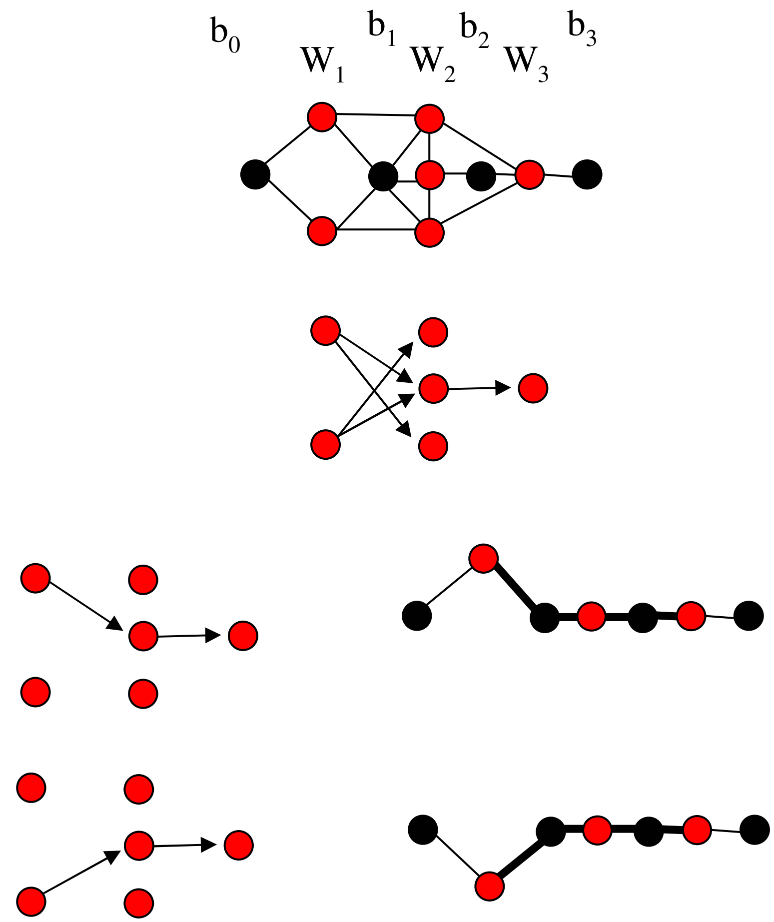
1. $k := 1$
2. Let b_k denote the second black neighbor of the vertices in W_k

   If b_k has 2 wings then go to 3.

   If b_k is irregular and different from b_0 then go to 4.

   Otherwise STOP (there is no pair (u,v))
3. Let W_k be the second wing of b_k; set $k := k+1$ and go to 2.
4. Construct an auxiliary graph with vertex set. $W_1 \cup \ldots \cup W_k$ and link two vertices if they are non-adjacent in G and belong to 2 consecutive W_i and W_{i+1}

   Orient all edges from W_i to W_{i+1}.
5. Determine the pairs (u,v) such that u is in W_1 and v in W_k and there is a path from u to v in the auxiliary graph.


b_0
W_1
b_1
W_2
b_2
W_3
b_3

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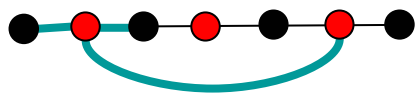
Property

The chain constructed by the previous algorithm has no short chord

Property

If G is claw-free then a white alternating chain without short chord is chordless



Corollary

The previous algorithm can detect all IWAPs

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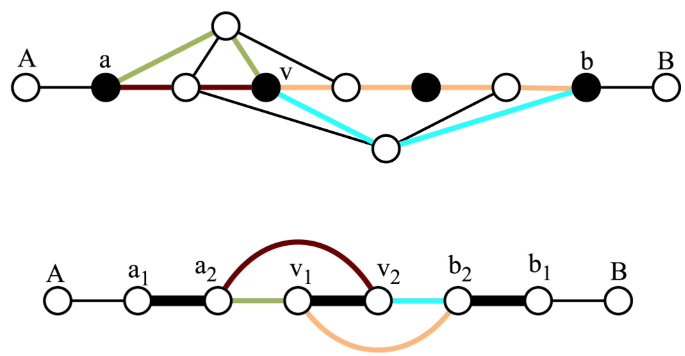
Edmond's graph

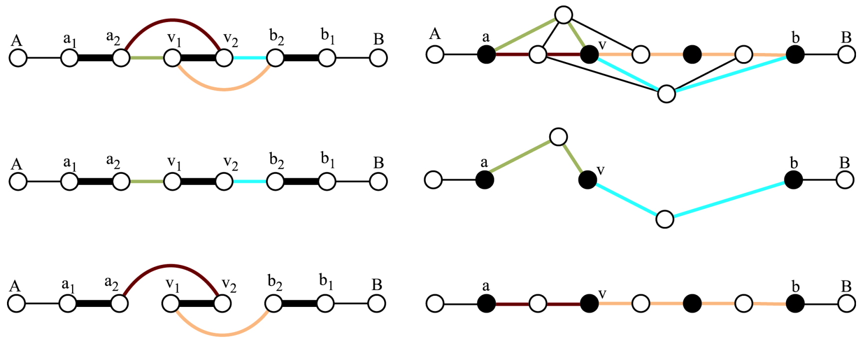
1. For each vertex v in R create two vertices v_1 and v_2 and link them by a black edge. Identify vertex v_1 with $N_1(v)$ and v_2 with $N_2(v)$
2. Create vertices A and B and link A to a_1 and B to b_1 by a white edge
3. Link v_i to w_j with a white edge if there exists an IWAP with termini x and y such that x is in $N_i(v)$ and y in $N_j(w)$. Identify the edge with such an IWAP



The black edges define a matching M in the Edmond's graph.

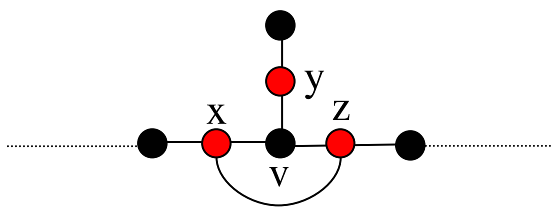
If M is not maximum then there exists an augmenting chain of edges.

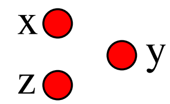
Such an augmenting chain in the Edmond's graph corresponds to an alternating chain in G



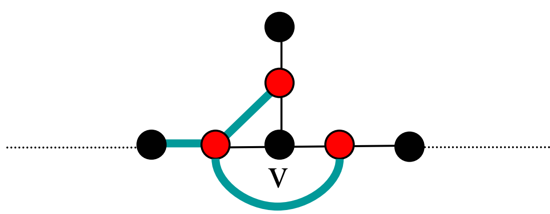
Theorem (Minty, 1980)

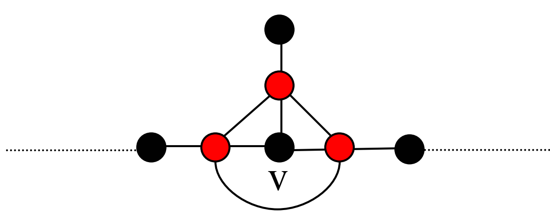
The above alternating chain has no short chord, which means that it is an augmenting chain

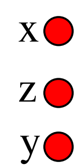




in H(v)







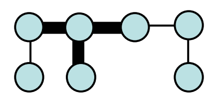
in H(v)

Minty's algorithm for finding augmenting chains in claw-free graphs

1. Partition the neighborhood of each regular black vertex v into two node classes by constructing the bipartite graph H(v) in which two white neighbors of v are linked by an edge if they are non-adjacent and non-similar.
2. Determine the pairs (u,v) of white vertices such that there exists an IWAP with termini u and v
3. Construct the Edmond's graph and let M be the set of black edges in it.
4. If the Edmond's graph contains an augmenting chain of edges with respect to M then it correspond to an augmenting chain in G. Otherwise there are no augmenting chains with termini A and B.

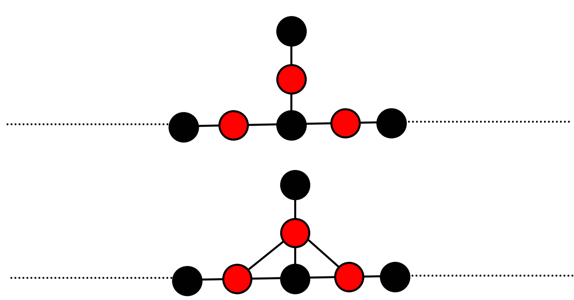
Augmenting Chains in $S_{1,2.3}$-free graphs

Gerber Hertz Lozin (2003)



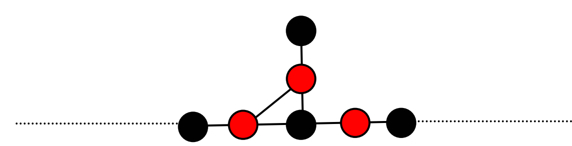
A pair (u,v) of vertices is special if u and v have a common black neighbor b and if there is a vertex w in N(b) which is similar neither to u nor to v and such that either both of uw and vw or non of them is an edge in G.

If (u,v) is a special pair of non-adjacent non-similar vertices in a $S_{1,2,3}$-free graph then u and v cannot occur in a same augmenting chain.



not ok

ok



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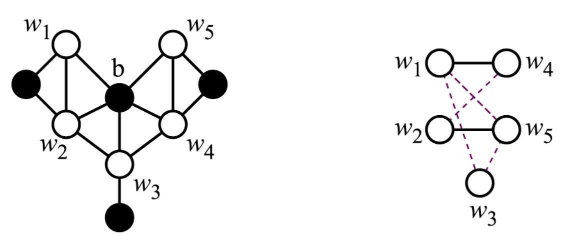
Consider the graph H(v) where

- The vertex set of H(v) is N(v)
- Two vertices u and w in H(v) are linked by an edge if and only if they (u,w) is a pair of non-similar, non-adjacent and non-special vertices

THEOREM (Gerber, Hertz, Lozin, 2003)

H(v) is bipartite for all regular vertices v



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Let $(b_1,w_1,\dots,w_{k-1},b_k)$ be an IBAP.

The IWAP $(w_1,\dots,w_{k-1})$ is interesting if w_1 and w_{k-1} are non-isolated in $H(b_1)$ and $H(b_k)$, respectively

Algorithm for finding augmenting chains in $S_{1,2,3}$ -free graphs

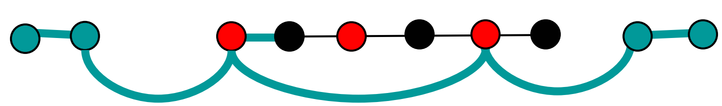
1. Partition the neighborhood of each regular black vertex v into two node classes by constructing the bipartite graph H(v) in which two white neighbors u and w of v are linked by an edge if (u,w) is a pair of non-adjacent, non-similar and non-special vertices.
2. Determine the pairs (u,v) of white vertices such that there exists an interesting IWAP with termini u and v
3. Construct the Edmond's graph and let M be the set of black edges in it.
4. If the Edmond's graph contains an augmenting chain of edges with respect to M then it correspond to an alternating chain in G. Otherwise there are no augmenting chains with termini A and B.

Theorem (Gerber, Hertz, Lozin, 2003)

The alternating chain found at Step 4 has no short chord

Corollary

The alternating chain found at Step 4 is an augmenting chain



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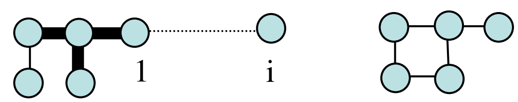
Augmenting Chains in ($S_{1,2,i}$,banner)-free graphs

Hertz Lozin Schindl (2003)

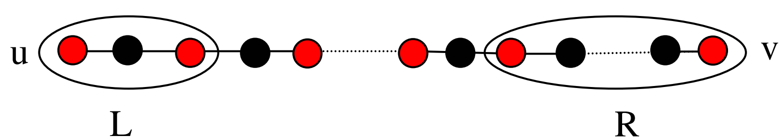


Let (u,v) be a pair of white non-adjacent vertices, each of which has exactly one black neighbor.

Let $m=2\lfloor i/2 \rfloor$

A pair (L,R) of disjoint chordless alternating chains $L=(u=x_0,x_1,x_2)$ and $R=(x_{k-m},x_{k-m+1},\ldots,x_{k-1},x_k=v)$ is candidate for (u,v) if

- no vertex in L is adjacent to a vertex of R
- each vertex x_j is white if and only if j is even



Augmenting chains with at most i+3 vertices can be found in polynomial time by inspecting all subsets of white vertices of cardinality (i+4)/2

Larger augmenting chains can be detected by applying the following algorithm to each pair (u,v) of white non-adjacent vertices, each of which has exactly one black neighbor

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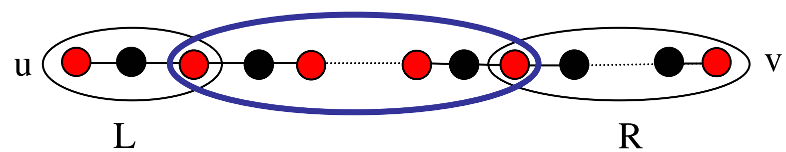
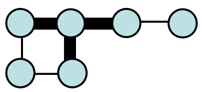
1. Remove from G all white vertices adjacent to u or v as well as all white vertices different from u and v which have 0,1 or more than 3 black neighbors
2. Find all candidate pairs (L,R) of alternating chains for (u,v) and for each such pair do the following
   - 2.1 remove all white vertices that have a neighbor in L or in R
   - 2.2 remove the vertices of L and R except x_2 and x_{k-m}
   - 2.3 remove all the vertices that are the center of a claw in the remaining graph
   - 2.4 in the resulting claw-free graph, determine whether there exists an augmenting chain with termini x_2 and x_{k-m}



Augmenting Complete Bipartite Graphs in $banner_2$-free graphs

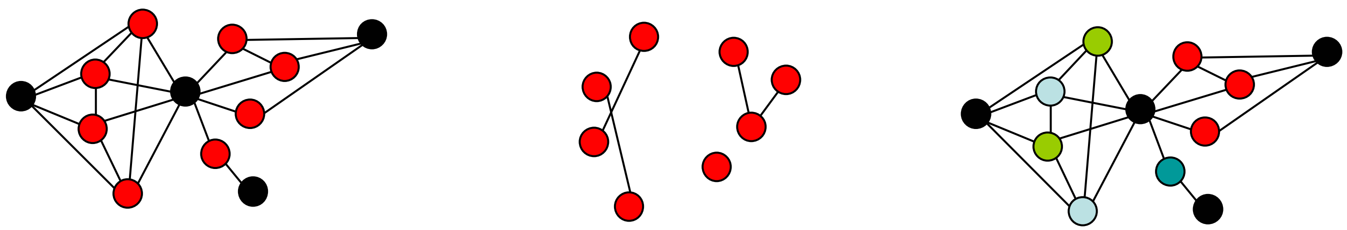
Hertz Lozin (2004)



Remember that two white vertices are similar if they have the same black neighbors

Let C be similarity class and let G[C] denote the subgraph induced by C.

Each subset of C that forms a maximal connected component in the the complement of G_C is called a Co-Class



We may assume that there is no augmenting $K_{1,2}$ and no augmenting $K_{2,3}$ since these augmenting graphs can easily be detected.

Hence, we may assume that each white vertex has at least 3 black neighbors.

We may also assume that each white vertex has at least one non neighbor in its co-class

Consider the following auxiliary weighted graph Γ

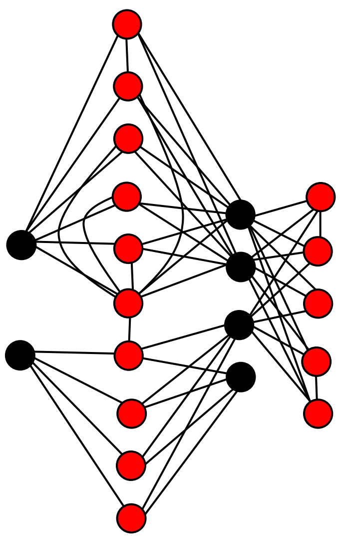
- The vertices of Γ are the node classes of G.
- Two vertices are non-adjacent if and only if the corresponding node classes Q_1 and Q_2 have no common black neighbor and no vertex in Q_1 is adjacent to a vertex in Q_2.
- The weight of a vertex Q is $\alpha(G[Q])-|N_S(Q)|$ where
  - $\alpha(G[Q])$ is the independence number of the subgraph induced by Q
  - $N_S(Q)$ is the set of black neighbors of the vertices in Q.

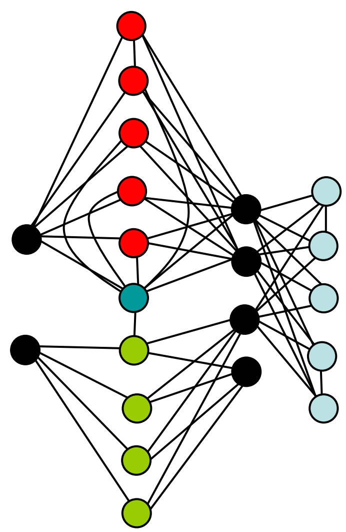
Recursive algorithm

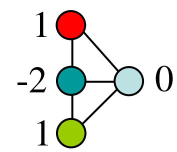
- Find an arbitrary maximal independent set S in G.
- If there is an augmenting graph H that contains a P_4 then augment S and repeat step 2.
- Partition the vertices of V-S into node classes $Q_1, \ldots, Q_k$
- Find a maximum independent set S_j in the graph induced by Q_j $(j=1,\ldots,k)$
- Construct the auxiliary graph Γ and find an independent set $Q=(Q_1,\ldots,Q_p)$ of maximum weight in it
- If the weight of Q is positive then the complete bipartite graph induced by

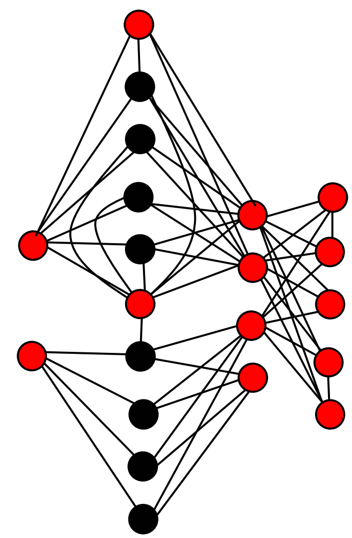
$$(N_S(Q_i) \cup \ldots \cup N_S(Q_p)) \cup (S_1 \cup \ldots \cup S_p)$$

is augmenting and by exchanging $N_S(Q_i) \cup \ldots \cup N_S(Q_p)$ with $S_1 \cup \ldots \cup S_p$ one get a maximum independent set in G.






1
-2
0
1



5 . find an independent set $Q=(Q_1,\ldots,Q_p)$ of maximum weight in Γ

Theorem (Hertz Lozin 2004)

If G is $banner_2$-free then

Γ contains no induced $K_{2,3}$, P_5 , C_5 , banner , and no odd anti –hole

Corollary

Γ is perfect

Hence, according to the result of Groetschel, Lovász and Schrijver (1984), an independent set of maximum weight in Γ can be found in polynomial time

Theorem

The maximum independent set problem in the class of $banner_2$-free graphs is polynomially equivalent to the problem of finding augmenting graphs containing a P_4.

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Theorem

Every minimal P_4-free augmenting graph is a complete bipartite graph

Theorem (Lozin, 2000)

Every minimal (P_5,banner)-free augmenting graph is a complete bipartite graph

Theorem (Hertz, Lozin, 2003)

Every minimal ($S_{1,2,2}$,banner)-free augmenting graph is either complete bipartite or a chain of even length

We now know that the MIS can be solved in polynomial time in these classes of graphs

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In what follows we denote

- W^i the set of white vertices having exactly i black neighbors
- B(w) the set of black neighbors of w

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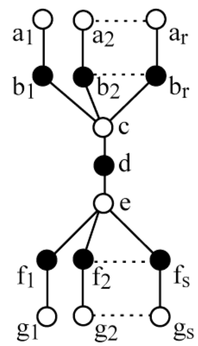
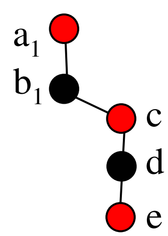
Finding an augmenting



Consider three white mutually non-adjacent vertices a_1,c,e such that

- $a_1 \in W^1$
- $|B(c)| \geq |B(e)|$
- $B(a_1) \cap B(c)=\{b_1\}$
- $B(c) \cap B(e)=\{d\}$
- $B(a_1) \cap B(e)=\emptyset$



The following algorithm determines whether this initial structure can be extended to the above augmenting graph

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1. Determine $A=(B(c) \cup B(e))-\{b_1,d\}$
2. For each vertex $u \in A$ determine the set $N_1(u)$ of white neighbors of u which are in W^1 and which are not adjacent to a_1, c or e
3. Let G' be the subgraph of G induced by $\cup_{u \in A} N_1(u)$
   - if $\alpha(G') = |A|$ then $A \cup \{a_1,b_1,c,d,e\}$ together with any maximum independent set in G' induce the desired augmenting graph
   - otherwise the initial structure cannot be extended to the desired augmenting graph.

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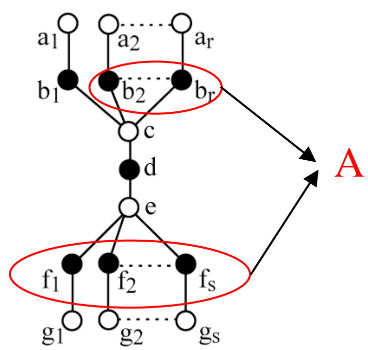

Theorem (Gerber, Hertz, Lozin, 2004)

If G is (P_8, banner)-free, then G’ is fork-free

Theorem (Alekseev, 1999)

The MIS can be solved in polynomial time in the class of fork-free graphs

Corollary

The above algorithm can be run in polynomial time

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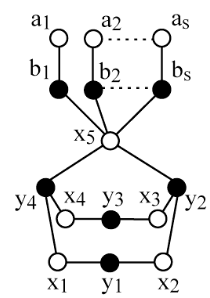
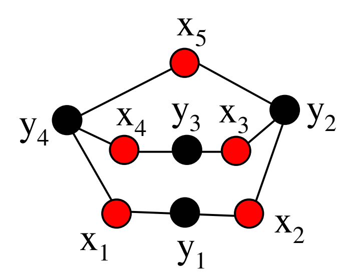
Finding an augmenting



Consider five white mutually non-adjacent vertices $x_1,x_2,\ldots,x_5$ such that

- $x_i \in W^2$ (i=1,2,3,4)
- $(x_1 \cup B(x_1)) \cup \ldots \cup (x_4 \cup B(x_4))$ is a $C_8(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)$
- $B(x_5) \cap \{y_1,y_2,y_3,y_4\} = \{y_2,y_4\}$



The following algorithm determines whether this initial structure can be extended to the above augmenting graph

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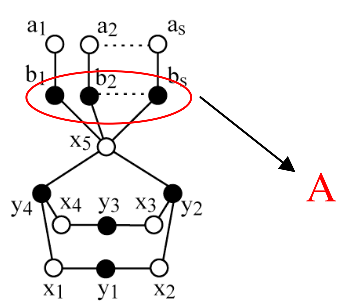
1. Determine $A=B(x_5)-\{y_2,y_4\}$
2. For each vertex $u \in A$ determine the set $N_1(u)$ of white neighbors of u which are in W^1 and which are not adjacent to x_1, x_2, x_3 or x_4
3. Let G' be the subgraph of G induced by $\cup_{u \in A} N_1(u)$
   - if $\alpha(G') = |A|$ then $A \cup \{x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3,y_4\}$ together with any maximum independent set in G' induce the desired augmenting graph
   - otherwise the initial structure cannot be extended to the desired augmenting graph.



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Theorem (Gerber, Hertz, Lozin, 2004)

If G is (P_8, banner)-free, then G' is the union of disjoint cliques

Corollary

The above algorithm can be run in polynomial time

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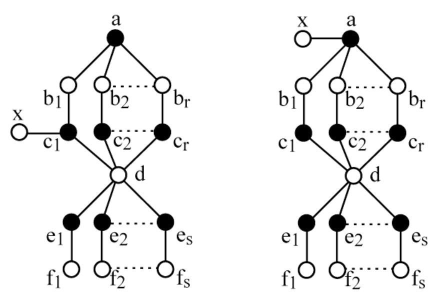
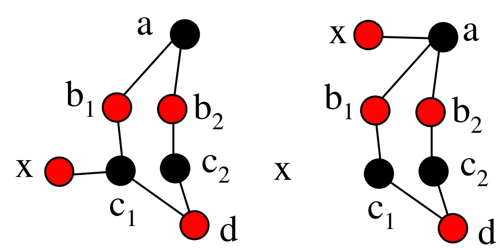
Finding an augmenting



Consider four white mutually non-adjacent vertices b_1,b_2,d,x such that

- $x \in W^1$
- b_1 and b_2 belong to W^2
- $\{b_1,b_2,d\} \cup B(b_1) \cup B(b_2)$ is a $C_6(c_1,b_1,a,b_2,c_2,d)$
- x is adjacent to a or (exclusive) c_1



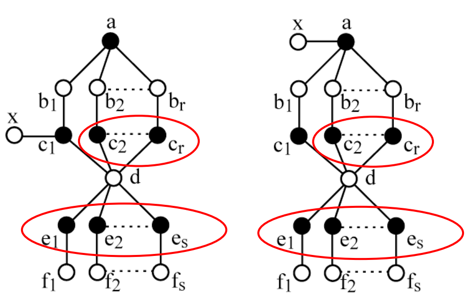
The following algorithm determines whether this initial structure can be extended to the above augmenting graph

1. Determine $A=B(d)-\{c_1,c_2\}$

2. For each vertex $u\in A$ determine the set $N_1(u)$ of white neighbors of u which are in W^1 and which are not adjacent to x, b_1, b_2 or d as well as the set $N_2(u)$ of white vertices in W^2 which are adjacent to both a and u but not to x, b_1, b_2 or d

2. Let G' be the subgraph of G induced by $\cup_{u\in A} (N_1(u) \cup N_2(u))$

   - if $\alpha(G')= |A|$ then $A\cup\{a,b_1,b_2,c_1,c_2,d,x\}$ together with any maximum independent set in G' induce one of the desired augmenting graph
   - otherwise the initial structure cannot be extended to the desired augmenting graph.



GERAD

ÉCOLE POLYTECHNIQUE MONTRÉAL

Theorem (Gerber, Hertz, Lozin, 2004)

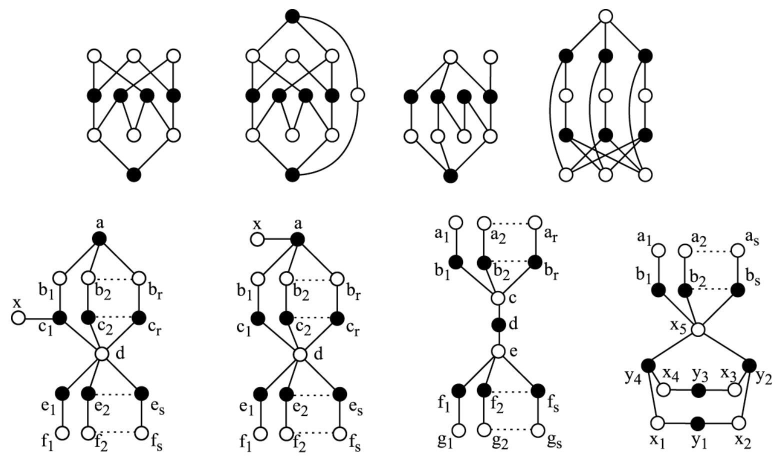
If G is (P_8, banner)-free, then G’ is fork-free

Corollary

The above algorithm can be run in polynomial time

Theorem (Gerber, Hertz, Lozin, 2004)

Every minimal (P_8,banner) -free augmenting graph is either complete bipartite or one of the following graphs



Corollary

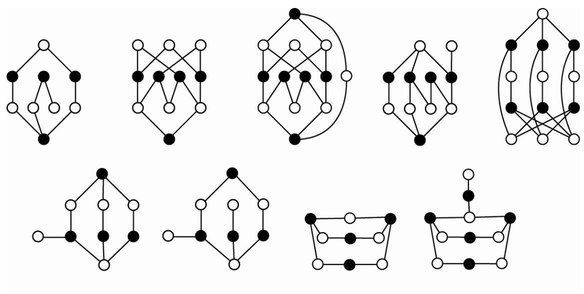
The MIS can be solved in polynomial time in the class of (P_8,banner)-free graphs

GERAD

ÉCOLE POLYTECHNIQUE MONTRÉAL

Theorem (Gerber, Hertz, Lozin, 2004)

Every minimal ($S_{1,2,4}$,banner) -free augmenting graph is either a complete bipartite graph or chain of even length, or a simple augmenting tree, or an augmenting plant or one of the following graphs.



Complete bipartite graph : OK since G is banner-free (Hertz, Lozin, 2004)

Chain of even length : OK since G is ($S_{1,2,4}$,banner)-free (Hertz, Lozin, Schindl, 2003)

Simple augmenting tree and augmenting plant : OK (Alekseev, Lozin, 2000)

The other 9 graphs : OK since they have at most 13 vertices

Corollary

The MIS can be solved in polynomial time in the class of ($S_{1,2,4}$,banner)-free graphs